Quantal Indeterminacy of Time

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1 Untitled

"QM applies to individual systems"	=	"There are no hidden variable theories compatible with QM."

i.e. it is probabilistic, but not statistical.

2 Untitled

Thus quantal systems do not 'have' values of their dynamical variables (unless they are in eigenstates), otherwise values are manifested with a certain 'spread' or spectral extension. A system, e.g., with spread in its position variable in the x-direction of Δx about some value x, is most probably found between $x - \frac{\Delta x}{2}$ and $x + \frac{\Delta x}{2}$.

Now the spectral extension is defined with respect to a given operator and a given wave-function $\psi(t)$, i.e., it is defined for a certain value of the time t. ψ on the other hand, is determined by the environment of the system at time t through the Hamiltonian operator. The environment being represented by the field potentials occurring in it.

The spread of x is defined through its square, the variance as

$$\begin{aligned}(\Delta_\psi x)^2 &\stackrel{\text{df}}{=} \langle\psi|(\hat{x})^2|\psi\rangle - |\langle\psi|\hat{x}|\psi\rangle|^2 \\ &= \|(\hat{x} - \langle\hat{x}\rangle)|\psi\rangle\|^2\end{aligned} \tag{1}$$

where $\hat{x}$ is the Hermitian operator representing the x-position variable. Similarly we may define

$$(\Delta_\psi p)^2 \stackrel{\text{df}}{=} \|(\hat{p} - \langle\hat{p}\rangle)|\psi\rangle\|^2 \tag{2}$$

$$\text{and} \qquad (\Delta_\psi E)^2 \stackrel{\text{df}}{=} \|(\hat{H} - \langle\hat{H}\rangle)|\psi\rangle\|^2 \tag{3}$$

Now Robertson's theorem states that

$$(\Delta_\psi A)^2(\Delta_\psi B)^2 \geq \tfrac{1}{4}|\langle[\hat{A},\hat{B}]\rangle_\psi|^2 \tag{4}$$

where $\hat{A}$ and $\hat{B}$ are the Hermitian operators representing the dynamical variables A and B resp. As is well known, Heisenberg relation between the variances of any two quantities represented by Hermitian operators is determinable from (4), in part:

$$\Delta_\psi p_x \Delta_\psi x \geq \tfrac{1}{2} \tag{5}$$

since

$$[\hat{p},\hat{x}] = -i\hbar \tag{6}$$

Now just as a system with a non-zero spectral extension of momentum eigenvalues Δp_x is not determinately realizable along the x-axis with a spread of less than $\Delta x = \frac{\hbar}{2\Delta p_x}$, so one would expect that a system not in an energy eigenstate would not be localizable in time with a spread of less than $\Delta t = \frac{\hbar}{2\Delta E}$.

Indeed, empirical considerations such as the relation between energy-spread and life-times of states of unstable systems, and line breadths in atomic spectrum, lead us to expect a relation similar to (5) between spreads of energy and temporal location.

$$\Delta_\psi E \Delta_\psi t \overset{?}{\geq} \tfrac{1}{2}\hbar \tag{7}$$

It should perhaps be emphasized at this point that there is nothing nonsensical about such a relation in principle, despite the occurrence of time as a scatter-free parameter t in quantum theory: so long as Δt is interpreted as the scatter of a dynamical variable representing the location of the system in state ψ along the time axis, the analogy with the position-momentum relation holds firm. The analogy would be perfect if relation (7) could be derived from (4) and a commutation relation between the Hamiltonian and a hermitian time operator $\hat{t}$ of the same kind as (6):

$$[\hat{\tau},\hat{H}] = +i\hbar \tag{8}$$

However, it is well known that because of the physical requirement that energy values (i.e. eigenvalues of $\hat{H}$) be only positive, it is not possible to construct a hermitian time-operator in the (real) energy representation which acts in the Hilbert space $L_2(0,\infty)$ of physical states. (This will be demonstrated below).

Undeterred by this lack of a hermitian operator canonically conjugate to H with which to represent dynamical time, many authors have attempted to derive the time-energy scatter inequality (7) by other means. All of them, however, (with the exception of Rankin, who works outside the normal quantum mechanical algorithm), either tacitly or explicitly define $\hat{\tau}$ (or $\Delta_\psi \tau$) on the quasi-energy space which has a continuous range of values ω from $-\infty$ to $+\infty$.

In fact it is fairly straightforward to define a hermitian time operator in this quasi-energy representation as $\hat{\tau}_\omega = i\hbar\frac{\partial}{\partial\omega}$, with $\hat{\omega}|\omega\rangle = \omega|\omega\rangle$, so that we obtain a commutation relation analogous to (8):

$$[\hat{\tau}_\omega, \hat{\omega}] = +i\hbar \tag{9}$$

from which a scatter relation follows immediately using (4)

$$\Delta\omega\Delta\tau_\omega \geq \tfrac{1}{2}\hbar \tag{10}$$

Now, it may be that ω can consistently be interpreted as virtual energy, as Almond proposes and Fujiwara assumes. However this still does not give us a time operator for real systems, nor a Heisenberg scatter relation between the spread of energy of a real system and its spectral extension in time.

If we consider the time operator in the quasi-energy representation but acting in the Hilbert space $L_2(0,\infty)$ of physical states, approximate eigenfunctions and eigenvalues can be found. Nowicki's thorough investigation has shown that in this case $\hat{\tau_\omega}$ takes the form

$$\hat{\tau_\omega} = \hat{\tau}_E = i\sqrt{\frac{2E}{m}}\frac{d}{dE}$$

and is symmetric on a domain $\mathcal{D}(\hat{\tau}_E)$ which is a dense subspace of $L_2(0,\infty)$.

The remainder of this paper is directed towards deriving the inequality (7) without assuming the existence of a Hermitian time operator satisfying (8) The standard derivation $\Delta_\psi\tau$ will be defined using a one-sided unitary operator, U_ω, whose eigenstates may be regarded as the time-eigenstates of the system. Although these eigenstates are necessarily non-orthogonal, the view is defended that they can nevertheless represent physical states.

The defense of this view, and indeed the entire method of treating the Heisenberg inequalities adopted here, were just proposed by J.M. Lévy-Leblond[1]. In fact, their extension to the time-energy case follows so naturally from this work that it is something of an enigma that he did not explicitly extend them to this case himself.

Lévy-Leblond argues that only two conditions are necessary on nonhermitian operators and nonorthogonal state vectors to preserve the usual probability interpretation. The first is a resolution of the identity for the state vectors $\{|a\rangle\}$:

$$1 = \int da|a\rangle\langle a| \tag{9}$$

The second is the existence of a function f such that the definition:

[1] "Who's Afraid of Nonhermitian Operators? A Quantum Description of Angle and Phase" Annals of Physics 101, No 1, Sept 1976

$$\widehat{f(a)} \stackrel{\mathrm{df}}{=} \int da\, f(a)|a\rangle\langle a| \tag{10}$$

yields a well-defined operator with the $|a\rangle$'s as eigenvectors. This condition is described by Lévy-Leblond as a "very stringent requirement," since if the $|a\rangle$'s are not mutually orthogonal it is by no means automatically fulfilled.

The decomposition (9) allows any normalized state vector $|x\rangle$ to be expanded as

$$|x\rangle = \int da|a\rangle\langle a|x\rangle \qquad \text{with} \qquad 1 = \int da|\langle a|x\rangle|^2$$

so that "the probabilistic interpretation of the $|\langle a|x\rangle|^2$ remains valid. Using (10), this means that the average value of the physical property with spectrum $\{f(a)\}$ is given by "the usual Hilbertian sand[?]"

$$\langle x|\widehat{f(a)}|x\rangle = \int da\, f(a)|\langle a|x\rangle|^2$$

3 Lévy-Leblond's Unitary Operator Method of Deriving Heisenberg Inequalities

Lévy-Leblond's method consists essentially in considering instead of a given Hermitian operator $\hat{a}$ the unitary operator generated by it: $\hat{U}_b = \exp \frac{ib}{\hbar}\hat{a}$. This method is then easily extendable to the cases where there is no Hermitian generator corresponding to the unitary operator in question.

Lévy-Leblond points out that it is false to assume that only Hermitian operators can be given a straightforward physical interpretation, citing the pervasive role played in[?] by unitary operators such as the S-matrix "and even non-normal ones, such as most field operators in quantum field theory ... That we keep calling "observables" only the Hermitian operators is but a lip-service paid to an obsolete orthodoxy, since no empirical or formal criterion exist (sic) to decide whether such or such an operator may actually be "observed."

As an example, it is well-known that the group of unitary operators

$$\hat{U}_p = \exp\left\{+\frac{ip}{\hbar}\hat{q}\right\} \tag{11}$$

constitute the group of translations in momentum space, such that

$$\psi(p+p') = \hat{U}_p\psi(p') \tag{12}$$

and

$$\hat{U}_p \, \hat{p} \, \hat{U}_p^\dagger = \hat{p} - p\hat{\mathbb{I}} \tag{13}$$

Since the hermiticity of $\hat{q}$ guarantees the unitarity of $\hat{U}_p$:

$$\hat{U}_p^\dagger \hat{U}_p = \hat{\mathbb{I}} \tag{14}$$

(13) may be written

$$[\hat{p}, \hat{U}_p] = +p\hat{U}_p \tag{15}$$

An expansion of $\hat{U}_p = \exp(+\frac{ip}{\hbar}\hat{q})$ and comparison of coefficients of terms of order p, p^2, p^3 gives the usual commutator

$$[\hat{p}, \hat{q}] = -i\hbar \tag{16}$$

so that (15) contains all the information of (16). Similarly, it is easily seen that $\hat{U}_p$ has the same eigenstates as $\hat{q}$:

$$\hat{q}|\bar{q}\rangle = \bar{q}|\bar{q}\rangle \Leftrightarrow \begin{array}{ll} \hat{U}_p|\bar{q}\rangle & = \exp\left(+\frac{ip}{\hbar}\bar{q}\right)|\bar{q}\rangle \\ \hat{U}_p^\dagger|\bar{q}\rangle & = \exp\left(-\frac{ip}{\hbar}\bar{q}\right)|\bar{q}\rangle \end{array} \tag{17}$$

Now we may apply the generalized version of Robertson's Theorem to derive a Heisenberg Inequality for $\hat{p}$ and $\hat{U}_p$, since $\hat{U}_p$ clearly satisfies the requisite conditions for a quantal probability interpretation:

$$\hat{U}_p \quad = \int dq \exp\left(+\frac{ip}{\hbar}q\right)|q\rangle\langle q| \tag{18a}$$

$$\hat{U}_p^\dagger \quad = \int dq \exp\left(-\frac{ip}{\hbar}q\right)|q\rangle\langle q| \tag{18b}$$

$$\text{and} \qquad \mathbb{I} = \int dq|q\rangle\langle q| \tag{19}$$

Using (15), Robertson's Theorem (A.10) gives

$$(\Delta_x p)^2(\Delta_x U_p)^2 \geq \tfrac{1}{4}p^2|\langle\hat{U}_p\rangle|^2 \tag{20}$$

From the definition of the variance of an operator (A.1)

$$(\Delta_x U_p)^2 = \langle x|U_p^\dagger U_p|x\rangle - |\langle U_p\rangle|^2 = 1 - |\langle U_p\rangle|^2 \tag{21}$$

An expansion of U_p in powers of p gives

$$
\begin{aligned}
(\Delta_x U_p)^2 = 1 - |\langle U_p \rangle_x|^2 &= 1 - \left| 1 + \frac{ip}{\hbar}\langle \hat{q} \rangle_x + \left(\frac{ip}{\hbar}\right)^2 \frac{1}{2!}\langle \hat{q}^2 \rangle_x + \mathcal{O}(p^3) \right|^2 \\
&= 1 - \left(1 + \frac{p^2}{\hbar^2}|\langle \hat{q} \rangle_x|^2 - \frac{p^2}{\hbar^2}\langle \hat{q}^2 \rangle_x + \mathcal{O}(p^4) \right) \\
&= \frac{p^2}{\hbar^2}\{\langle \hat{q}^2 \rangle_x - |\langle \hat{q} \rangle_x|^2\} + \mathcal{O}(p^4) \\
&= \frac{p^2}{\hbar^2}(\Delta_x q)^2 + \mathcal{O}(p^4)
\end{aligned}
$$

We thus obtain the expression

$$
\Delta_x q = \lim_{k \to 0}\left(\frac{1}{k}\Delta_x U_p\right) \tag{22}
$$

Now combining (20) and (21) gives

$$
\begin{aligned}
(\Delta_x p)^2(\Delta_x U_p)^2 &\geq \frac{p^2}{4}(1 - (\Delta_x U_p)^2) \\
\text{or} \qquad (\Delta_x p)^2\left(\frac{\hbar}{p}\Delta_x U_p\right)^2 &\geq \frac{\hbar^2}{4} - \frac{p^2}{4}\left(\frac{\hbar}{p}\Delta_x U_p\right)^2
\end{aligned} \tag{23}
$$

Thus is we take the limit of each side of (23) as $p \to 0$, and use (22) as our definition of the spread of $\Delta_x q$, we regain the normal Heisenberg inequality for p and q,

$$
(\Delta_x p)(\Delta_x q) \geq \tfrac{\hbar}{2} \tag{24}
$$

Let us now consider the case of energy and localization in time.

The unitary operator for translations in energy space, $\hat{U}_E$, satisfies

$$
\psi(E + E') = \hat{U}_E \psi(E') \tag{25}
$$

and

$$
\hat{U}_E \hat{H} \hat{U}_E^\dagger = \hat{H} - E\hat{\mathbb{I}} \tag{26}
$$

However, as was noted on [p2], the Hamiltonian $\hat{H}$ has no hermitian canonical conjugate $\hat{\tau}$, i.e. $\hat{U}_E$ has no canonical hermitian generator. This is because of the boundedness of the spectrum of E: although $\hat{U}_E$ successfully boosts states and operators by an amount $+E$, so that $\psi(E)$ runs from $\psi(0)$ to $\psi(+\infty)$, $\hat{U}_E^\dagger$ is not its inverse ($\hat{U}_E^{-1} \neq \hat{U}_E^\dagger$). An $\hat{U}_E^{-1}$ would boost states by $-E$ indefinitely, whereas there are no physical states $\psi(E)$ with $E < 0$. In fact, $\hat{U}_E$ is an example of a "one-sided unitary operator". H and its adjoint satisfy

$$\hat{U}_E^\dagger \hat{U}_E = 1 \tag{27}$$

although

$$\hat{U}_E \hat{U}_E^\dagger \neq 1 \tag{28}$$

But not, if we look back through the 'unitary method' derivation of (24), we see that only the one-way relation

$$\hat{U}_p^\dagger \hat{U}_p = 1 \tag{14}$$

is needed for the derivation of equations (15), (21). This means that if we take equations (25)–(28) to define $\hat{U}_E$, and then define a time-localization indeterminacy $\Delta_x t$ analogously to equation (22):

$$\Delta_x t \stackrel{\text{df}}{=} \lim_{E\to 0}\left(\frac{\hbar}{E}\Delta_x U_E\right). \tag{29}$$

We can readily obtain the desired energy-time inequality. From (26) and (27) we obtain the commutator between $\hat{H}$ and $\hat{U}_E$

$$[\hat{H}, \hat{U}_E] = +E\hat{U}_E, \tag{30}$$

analogous to that between $\hat{p}$ and $\hat{U}_p$ (15). Now, despite the absence of an analogue to (16)

$$[\hat{H}, \hat{t}] = +i\hbar \qquad \text{(wrong)}, \tag{31}$$

we may apply the generalized Robertson's Theorem to (30). Thus, using definition (29) and with the energy spread of the system defined as usual by

$$\Delta_x E \stackrel{\text{df}}{=} \sqrt{(\langle \hat{H}^2\rangle - \langle \hat{H}\rangle^2)} \tag{32}$$

we obtain

$$\boxed{\Delta_x E \Delta_x t \geq \tfrac{1}{2}\hbar,} \tag{33}$$

our desired result.

To recapitulate, this derivation depends only on the existence of the one-sided unitary operator $\hat{U}_E$ which generates energy translations and satisfies relations (25)–(28). The one-sidedness of $\hat{U}_E$ corresponds to the one-sidedness of the energy spectrum, assumed continuous from 0 to ∞ – (Actually, this proof is equally valid for the discrete case, with energy values $E_n, n = 1 \to \infty$.) – and accounts for the absence of a canonical hermitian generator. We may further investigate this one-sided unitary character of $\hat{U}_E$ as follows. Equations (25) and (27) give

$$\hat{U}_E|E'\rangle = |E+E'\rangle \qquad \text{and} \qquad \hat{U}_E^\dagger|E+E'\rangle = |E'\rangle \tag{34}$$

Thus

$$\hat{U}_E|\psi\rangle = \int dE'\hat{U}_E|E'\rangle\langle E'|\psi\rangle = \int dE'|E+E'\rangle\langle E'|\psi\rangle$$
$$\Rightarrow \hat{U}_E \equiv \int dE'|E+E'\rangle\langle E'| \tag{35}$$

Similarly,

$$\hat{U}_E^\dagger \equiv \int dE'|E'\rangle\langle E'+E| \tag{36}$$

These equivalences give

$$\hat{U}_E^\dagger\hat{U}_E = \int_0^\infty dE'U_E^\dagger|E+E'\rangle\langle E'| = \int dE'|E'\rangle\langle E'| = \hat{\mathbb{I}}, \tag{27}$$

as expected, and

$$\begin{aligned}\hat{U}_E\hat{U}_E^\dagger &= \int_0^\infty dE'U_E|E'\rangle\langle E'+E| = \int_0^\infty dE'|E+E'\rangle\langle E'+E| \neq \hat{\mathbb{I}} \\ &= \int_0^\infty d(E+E')|E+E'\rangle\langle E+E'| - |E_0\rangle\langle E_0| \\ &= \hat{\mathbb{I}} - \hat{P}_0 \end{aligned} \tag{37}$$

where $\hat{P}_0 = |E_0\rangle\langle E_0|$ is the projector onto the ground state.

Now,

$$\begin{aligned}\hat{U}_E^\dagger\hat{U}_E\hat{U}_E^\dagger &= \hat{U}_E^\dagger - \hat{U}_E^\dagger|E_0\rangle\langle E_0| \\ \Rightarrow \hat{U}_E^\dagger &= \hat{U}_E^\dagger - \hat{U}_E^\dagger|E_0\rangle\langle E_0| \qquad \text{using (27)}\end{aligned}$$

so that

$$\hat{U}_E^\dagger|E_0\rangle = 0 \tag{38}$$

– again in accordance with physical expectations.

However, the existence of $\hat{U}_E$ and $\hat{U}_E^\dagger$, and the validity of the above proof of (33), depend explicitly on the applicability to $\hat{U}_E$ and its adjoint of quantal probability considerations. Thus $\hat{U}_E$, analogously to $\hat{U}_p$ in equations (18) and (19),
on a basis of 'time eigenstates' $\{`|t\rangle'\}$:

$$\hat{U}_E \stackrel{?}{=} \int_{-\infty}^{\infty} dt \exp\Big(+\frac{iEt}{\hbar}\Big)`|t\rangle\text{'}`\langle t|\text{'} \tag{39a}$$

$$\hat{U}_E^\dagger = \int_{-\infty}^{\infty} dt \exp\Big(-\frac{iEt}{\hbar}\Big)`|t\rangle\text{'}`\langle t|\text{'} \tag{39b}$$

$$\text{with} \qquad 1 \stackrel{?}{=} \int_{-\infty}^{\infty} dt`|t\rangle\text{'}`\langle t|\text{'} \tag{40}$$

Let us now turn to a consideration of these 'time eigenstates' and whether they allow of a reasonable physical interpretation.

4 Time Eigenstates

The effective or formal existence of time-eigenstates '$|t\rangle$' is easily demonstrated independently of the question of the existence of the group of one-sided unitary operators U_E considered above. In fact, their formal existence is equivalent to that of the effective transformation function

$$`\langle E|t\rangle\text{'} \equiv h^{-\frac{1}{2}} \exp\Big\{-\frac{itE}{\hbar}\Big\} \tag{41}$$

exploited by Fujiwara and Durand in their (independent) attempts to establish the inequality (33), as we shall see below.

Dirac first derived the p-q transformation

$$\langle p|q\rangle = h^{-\frac{1}{2}} \exp\Big(-\frac{ipq}{\hbar}\Big) \tag{42}$$

by exploiting the definition

$$\hat{p} \stackrel{\text{df}}{=} -i\hbar\frac{\partial}{\partial q} \tag{43}$$

which satisfies the p–q commutation relation (16). However, it could just as easily be derived using instead of (43) the unitary operator

$$\psi(q+q') = \hat{U}_q\psi(q'), \qquad \hat{U}_q \stackrel{\text{df}}{=} \exp\Big\{-\frac{iq\hat{p}}{\hbar}\Big\} \tag{44}$$

which effects translations in configuration space. Thus

$$\psi(q) = \hat{U}_q|\psi\rangle = \int_{-\infty}^{\infty} \exp\Big\{-\frac{iq\hat{p}}{\hbar}\Big\}|\bar{p}\rangle\langle\bar{p}|\psi\rangle dp', \tag{45}$$

where $|\psi\rangle$ is the initial state, on expanding in terms of the orthonormal momentum eigenstates $\{|p'\rangle\}$. Setting $\psi(\bar{p}) = c|\bar{p}\rangle\langle\bar{p}|\psi\rangle$, the projection of $|\psi\rangle$ onto the eigenstate $|p'\rangle$, with c the normalization constant, and noting that

$$\exp\Big(-\frac{iq\hat{p}}{\hbar}\Big)|p'\rangle = \exp\Big\{-\frac{iqp'}{\hbar}\Big\}|p'\rangle,$$

we obtain

$$\psi(q) = \int_{-\infty}^{\infty} \frac{1}{c}\exp\Big(-\frac{iqp}{\hbar}\Big)\psi(p)dp \tag{46}$$

Now, since Fourier analysis yields $|c|^2 = 2\pi\hbar$, and the transformation function $\langle p|q\rangle$ is defined by

$$\psi(q) = \int_{-\infty}^{\infty} \psi(p)\langle p|q\rangle dp \tag{47}$$

comparison of (46) and (47) yields the p-q transformation function (42).

Similarly, the Schrödinger Equation enables us to define a unitary operator

$$\hat{U}_t \stackrel{\text{df}}{=} \exp\Big(-\frac{it\hat{H}}{\hbar}\Big) \tag{48}$$

so that we have, with $\hat{H}|E\rangle = E|E\rangle$, $\int dE\,|E\rangle\langle E| = 1$ and $\psi(E) \stackrel{\text{df}}{=} h^{\frac{1}{2}}|E\rangle\langle E|\psi\rangle$,

$$\begin{aligned}\psi(t) = \hat{U}_t|\psi\rangle &= \int_0^{\infty} \exp\Big(-\frac{it\hat{H}}{\hbar}\Big)|E\rangle\langle E|\psi\rangle \\ &= \int_0^{\infty} h^{-\frac{1}{2}}\exp\Big(-\frac{it\hat{E}}{\hbar}\Big)\psi(E)dE \end{aligned} \tag{49}$$

Thus if $\langle E|t\rangle$ is defined by

$$\psi(t) = \int_0^{\infty} \psi(E)\langle E|t\rangle dE, \tag{50}$$

we obtain

$$\langle E|t\rangle = h^{-\frac{1}{2}}\exp\Big(-\frac{itE}{\hbar}\Big). \tag{41}$$

This transformation function gives us the means to make canonical transformations from the basis $\{|E\rangle\}$ to $\{|t\rangle\}$. Multiplying both sides by $|E\rangle$ and integrating, we obtain

$$|t\rangle = h^{-\frac{1}{2}}\int_0^{\infty} \exp\Big(-\frac{itE}{\hbar}\Big)|E\rangle dE, \tag{51}$$

which we may take as our definition of $|t\rangle$.

We may now use this definition of $|t\rangle$ to establish this viability of equations (39) and (40). That the time eigenstates allow a resolution of the identity is easily shown:

$$\begin{aligned}\int_{-\infty}^{\infty}|t\rangle\langle t|dt &= \frac{1}{h}\int_{-\infty}^{\infty}dt\int_{0}^{\infty}dE\int_{0}^{\infty}dE'\exp\Big\{-\frac{it}{\hbar}(E-E')\Big\}|E\rangle\langle E'| \\ &= \frac{2\pi\hbar}{h}\int_{0}^{\infty}|E\rangle\langle E|dE = 1 \qquad (40)\end{aligned}$$

Similarly, on substitution of (51) in the definitions (39a) and (39b) of $\hat{U}_E$ and $\hat{U}_E^{\dagger}$ in terms of time eigenstates, these expressions reduce to the energy eigenstate expressions (35) and (36). However, although the $\{|t\rangle\}$ are eigenstates of $\hat{U}_E^{\dagger}$, as we would expect:

$$\begin{aligned}\hat{U}_E^{\dagger}|t\rangle &= \int_{0}^{\infty}dE'|E'\rangle\langle E'+E|t\rangle \qquad \text{(using (36))} \\ &= \int_{0}^{\infty}dE'\exp\Big\{-\frac{it}{\hbar}(E'+E)\Big\}|E'\rangle \qquad \text{(using (41))} \\ &= \exp\Big(-\frac{itE}{\hbar}\Big)|t\rangle \qquad (52)\end{aligned}$$

surprisingly, they are not eigenstates of $\hat{U}_E$:

$$\begin{aligned}\hat{U}_E|t\rangle &= \int_{0}^{\infty}dE'|E'+E\rangle\langle E'|t\rangle \\ &= \int_{0}^{\infty}dE'|E+E'\rangle\exp\Big(-\frac{itE'}{\hbar}\Big) \\ &= \exp\Big(\frac{itE}{\hbar}\Big)\int_{E}^{\infty}|E''\rangle\exp\Big(-\frac{iE''t}{\hbar}\Big)dE'' \qquad (E''=|E+E'\rangle) \\ &= \exp\Big(\frac{itE}{\hbar}\Big)\{|t\rangle-|E_0\rangle\langle E_0|t\rangle\} \qquad \text{(compare (37))} \\ &= \exp\Big(\frac{itE}{\hbar}\Big)(\hat{\mathbb{I}}-\hat{P}_0)|t\rangle \qquad (53)\end{aligned}$$

Whether this creates insurmountable problems for a physically meaningful interpretation of the $\{|t\rangle\}$ we will consider below. But first we should draw attention to the most obvious difficulty with the time-eigenstates: their lack of mutual orthogonality:

$$\langle t'|t\rangle = \frac{1}{h}\int_{0}^{\infty}\exp\Big\{-\frac{iE}{\hbar}(t-t')\Big\}dE \neq \delta(t-t')$$

5 Physical Interpretation of the Time Eigenstates