On the mathematization of free fall: Galileo, Descartes, and a history of misconstrual[1]

[1] This chapter grew out of research conducted in the Fisher Rare Books Library at the University of Toronto in 2001-02. Previous versions of it were read in Dubrovnik (31st *International Conference on the Philosophy of Science* April 2004), Winnipeg (*Canadian Society for the History and Philosophy of Science*, Annual Meeting, June 2004), Pisa (workshop on the Contested Expanding Rôle of Applied Mathematics from the Renaissance to the Enlightenment, September 2010), and London, Ontario (at the *Workshop on the Language of Nature* at the Rotman Institute, October, 2012). I am grateful to my audiences there for their comments, and especially to Geoffrey Gorham and the other participants in the Language of Nature workshop for their helpful comments and advice.

1. INTRODUCTION

In any attempt to understand conceptual history, the cardinal rule is *not* to assume that thinkers in the past were trying to express what we now understand with perfect clarity. For it will almost always turn out both that their understanding was different from ours—that apparently innocuous details like using proportions instead of equations "shift" their whole understanding with respect to ours—and that "we", in any case, do not understand the matter as perfectly as we would like to believe. Now, if we have learned this lesson of humility, it is partly from reading the classic studies of Alexandre Koyré, E. A. Burtt, Cornelis de Waard, Marshall Clagett, Thomas Kuhn, Stillman Drake, and other doyens of the history of science, and partly from finding them violating that rule themselves. For it is a principle that is impossible to apply in a total sense: there are always aspects of our understanding of a historical problem that we are unwittingly projecting onto past thinkers where we should not. So if in what follows I should appear to be hard on earlier historians of science, this must constantly be borne in mind; I do not pretend to be free of the vice myself.

The case I want to discuss here is the understanding of motion in the first half of the seventeenth century, specifically the case of the mathematization of free fall. Galileo famously established in his *Discorsi* that the distances traversed by a heavy body falling from rest in successive equal times are as the odd numbers 1, 3, 5, 7, …, or equivalently, that the total distances fallen are proportional to the squares of the times of fall (I shall refer to this as Galileo's "law of fall" or "t^2 law"[2]). Given the impact that the Galilean law of fall had on the mathematization of physics in the hands of Huygens, Leibniz and Newton, this mathematical model has come to be regarded as a constitutive element in the new paradigm of natural philosophy known as the Mechanical Philosophy. Kuhn's term "paradigm" has of course been almost voided of content by its initial ambiguity and its subsequent constant overuse and misapplication. But here, one might suppose, we are on safer ground: the mathematical modeling of free fall is an *exemplar*

[2] Here "law" should not be understood to connote a universal law of nature—see Daniel Garber's chapter in this volume.

that featured large in Huygens' *Pendulum Clock*, in Leibniz's derivation of the measure of *vis viva* as proportional to v^2, and in Newton's derivation of his inverse square law of gravitation.

It is also standardly supposed that the way of modeling motion implicit in this exemplar may be unproblematically ascribed to Galileo (and to a lesser extent, Descartes). On this conception, the trajectory of a moving body is represented by a curve on a graph of space traversed against time elapsed, and the body has at every instant of its motion an instantaneous velocity that is a function of time elapsed, and whose magnitude is given by the slope of the tangent to the curve at that instant. Such an understanding is usually ascribed to the original efforts of these thinkers themselves. For it is to Galileo that we owe the geometric representation of the curved trajectory of a body in motion, and to Descartes the expression of the curve as an algebraic equation, with both thinkers resolving such motion into orthogonal components; while the concept of instantaneous velocity derives from the notion of a degree of speed used by Galileo in his analysis of uniform acceleration, as well as from Descartes' notion of *conatus*.

Nevertheless, I shall argue, Galileo, Descartes and others did not yet have our modern understanding of motion as a function of instantaneous velocity, since velocity for them was an affection of motion, and there is no motion in an instant.[3] The initial evidence for this is in the form of paradoxes and incongruities that arise from historians of science projecting this modern understanding back onto those authors, and the unconvincingness of their attempts to attribute the resulting confusions to the early natural philosophers themselves. This prompts the question, how was motion conceptualized prior to the modern account involving instantaneous velocity? As soon as this question is raised, the initial appearance of a clear exemplar of mathematization of motion begins to evaporate; that is not to say that such an exemplar does not eventually emerge, but that the process was nowhere near as smooth as it would appear from our projections of the modern understanding back onto its originators.

[3] As we shall see below, much the same point has been made by Damerow et al. (1991), and by Jullien and Charrak (2002). My thanks to Dan Garber for drawing my attention to the latter during the workshop.

There were, I contend, several strands in early seventeenth century thinking about motion, which variously complemented or contradicted one another. There existed a strong presumption from Aristotelian philosophy that motion, like time, is continuous. But there was also a widespread conviction that changes in motion occurred discontinuously, with increases in the velocity of motion occurring by the addition of discrete increments of uniform motion—a discretist conception of acceleration that Stillman Drake refers to (rather unhappily) as "the quantum theory of speeds". There was also a second non-continuist model of motion as consisting in an alternation of motions and periods of rest, with differences in speed accounted for in terms of different proportions of motions and rests, associated in particular with Arriaga (1632). Then there was Galileo's model of non-uniform motion, indebted to the Scholastic theory of intension and remission of forms, according to which acceleration occurs "continuously from moment to moment, and not interruptedly from one quantified part of time to another"[4], as the moving body goes through an actual infinity of degrees of speed which, taken together, constitute its "overall velocity". This theory ran headlong into the notorious difficulties of the composition of the continuum, a fact which provided continued motivation for the two discretist models just mentioned.

Now, one could characterize this state of affairs by saying that, prior to the advent of the functional model of motion established in the eighteenth century, the understanding of motion was in a state of Kuhnian crisis: there was no universally accepted paradigm, and instead several competing theories or paradigms, none of which commanded universal assent. I am not convinced that that is the best way to conceive things, since it suggests that there were definite, well-formed theories or factions in competition with one another. Yet there was not, for example, an Aristotelian theory of non-uniform motion, nor was there a "quantum theory of speeds" in the sense of a clearly articulated theory. In the last section of this paper I will revisit this issue, and suggest a different way of characterizing the situation. But before we

[4] "Ma perchè l'accelerazione si fa continuamente di momento in momento, e non intercisamente di parte quanta di tempo in parte quanta…" (*Dialogo*; Galilei 1897, 162). All translations given in this paper from the Latin, Italian and French are my own. English translations of all relevant texts may also be found in Damerow et al. (1991).

can intelligently discuss such historiographical matters, we need to examine the specifics of the case before us, the mathematization of free fall by Galileo and his predecessors, and by Descartes.

2. GALILEO AND HIS PREDECESSORS

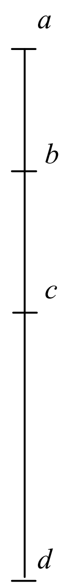
Let us begin with Galileo's criticism in the *Discorsi* of his predecessors' view that a falling body will move more swiftly as its distance from its point of origin increases. As is well known, Galileo had arrived at the correct law for freely falling bodies in both its forms already in 1604. In a letter to his friend Paolo Sarpi of 16th October, 1604, he writes that "*spaces traversed in natural motion are in the double proportion of the times*"—the t^2 rule for the proportion of distance to time of fall—"and consequently that *the spaces traversed in equal times are as the odd numbers starting from 1*"[5]—the odd-number rule for successive distances covered in successive times. But in the same letter he announces the principle on which this law is based:

> And the principle is this, that the natural moving body goes by increasing its velocity in the proportion that it is distant from the beginning of its motion; as, for example, with a heavy body falling from the point *a* through the line *abcd*, I suppose that the degree of velocity which it has in *c* to the degree it has in *b* is as the distance *ca* is to the distance *ba*, and so consequently in *d* it will have a degree of velocity more than in *c* according as the distance *da* is more than *ca*.[6]

[5] "… cioè gli spazzii passati dal moto naturale esser in proporzione doppia dei tempi, et per conseguenza gli spazii passati in tempi eguali esser come i numeri impari *ab unitat*e" (*Opere* X, p. 93)

[6] "Et il principio è questo: che il mobile natural vadia crescendo di velocità con quella proportione che si discosta dal principio del suo moto; come, v.g., cadendo il grave dal termine *a* per la linea *abcd* suppongo che il grado di velocità che ha in *c* al grado di velocità che hebbe in *b* esser come la distanza *ca* alla distanza *ba* et così conseguentemente in *d* haver grado di velocità maggiore che in *c* secondo che la distanza *da* è maggiore della *ca*."

Figure 1: Galileo to Sarpi, 1604



We tend automatically to interpret Galileo's "degree of velocity" as an instantaneous velocity, and therefore take his principle to be that the velocity of fall at a given instant is proportional to the distance through which the body has fallen from rest. This is how the principle was interpreted by Koyré, following accounts previously given by Paul Tannery and Ernst Mach. But then "the correct formula for the law '*the speed of the moving body is proportional to the distance covered*' would be an exponential function", writes Koyré, citing Tannery (1926, 441 ff.). The argument for this claim is sketched "in our language of today" by Mach, and can be filled out as follows.[7] If the instantaneous velocity v is related to distance fallen s in a given time t by

$v = \mathrm{d}s/\mathrm{d}t = as$, with a constant, then

$\int \mathrm{d}s/s = \int a \, \mathrm{d}t = \int \mathrm{d}s/s$, and

$\log_e s = at + c$[8]

[7] See Ernst Mach, *Die Mechanik* (1973, 245-6); English translation in Mach 1902, 247-248.
[8] The logarithms can be taken to any base, but here, as throughout this paper, I will take them to base e (Naperian logarithms), which, for visual clarity, I write $\log_e x$ rather than the usual $\ln x$.

Here there is a difficulty that if $s = 0$ at $t = 0$, we obtain $c = \log_e 0$. But $\log_e 0$ is undefined. This can be circumvented if instead we assume that the body is already in motion at $t = 0$. For if at $t = 0$ we have $s = A$ (and thus $v = aA$), with A another constant, we obtain $c = \log_e A$, so that

$$\log_e s - \log_e A = \log_e (s/A) = at$$

giving

$$s = A \exp(at)$$

and

$$v = \mathrm{d}s/\mathrm{d}t = aA \exp(at)$$

Thus both the distance and the velocity increase exponentially with time elapsed.

Galileo soon realizes his error (most probably by 1610, although we have no direct documentary evidence), and in the *Dialogo* of 1632 correctly characterizes the degrees of velocity (represented by transverse lines) as increasing in length in proportion to increasing time, not to distance fallen. As he expresses it in the *Discorsi*, "in equal time intervals, the body receives equal increments of velocity;… the acceleration continues to increase according to the time and duration of motion".[9] At this point in the dialogue Galileo has Sagredo say that "it seems to me that this could be defined with perhaps greater clarity without altering the conception as follows: a uniformly accelerated motion is that in which the velocity will have increased in proportion to the increase in the space that has been traversed".[10] This affords Galileo the opportunity to admit (through Salviati) that this is how he once conceived things, and to offer an argument to show why this proposition is not a clearer or even an equivalent way of describing the case, but is in fact "false". (We will come back to that argument presently.) He proposes instead the following principle:

[9] "in tempi eguali si facciano eguali additamenti di velocità; …l'accelerazione loro vadia crescendo secondo che cresce il tempo e la durazione del moto." (*Discorsi*, Third Day; Galilei 1898, 202)
[10] "mi pare che con chiarezza forse maggiore si fusse potuto definire, senza variare il concetto: Moto uniformemente accelerato esser quello, nel qual la velocità andasse crescendo secondo che cresce lo spazio che si va passando." (*ibid*.)

> We call that motion equably or uniformly accelerated which, starting from rest, acquires equal moments of velocity in equal times.[11]

Now Mach[12] and company interpret this principle as stating that the instantaneous velocity increases in proportion to the time, so that $dv = g\,\mathrm{d}t$, where g is a constant, giving

$$v = \mathrm{d}s/\mathrm{d}t = gt$$

Now,

$s = \int g\,t\,\mathrm{d}t$ and, with $s = 0$ at $t = 0$, the t^2 law correctly follows:

$$s = {}^{1}/_{2}\,g\,t^2$$

The question now arises, how is it that Galileo could have identified the wrong principle in the first place? Here it must be admitted that the fallacy is a subtle one, which is hard to see without the benefit of the calculus. If no time has elapsed, the falling body has no speed and has fallen through zero distance; then as it falls, its speed increases as time elapses, but so does the distance it covers. It goes faster the further it is from its starting point, and it does not seem to matter whether the remoteness from its starting point is thought of temporally or spatially. But this does not explain why Galileo chooses initially to enunciate the principle in terms of space covered.

One explanation offered by historians of science (Koyré cites Emil Wohlwill and Pierre Duhem) is that in so doing, Galileo was simply drawing on the tradition. For Leonardo da Vinci had written that "The heavy body in free fall acquires a degree of motion with each degree of time, and a degree of speed with each degree of motion".[13] Yet he had still "asserted that the speed is proportional not to the time elapsed but to the distance covered", as had Giovanni Battista Benedetti and Michel Varron, the former of whom was certainly an influence on Galileo (Koyré 1978, 70). But Koyré rejects the sufficiency of this

[11] "Moto equabilmente, ossia uniformemente accelerato, diciamo quello che, a partire dalla quiete, in tempi eguali acquista eguali momenti di velocità." (*Discorsi*, Third Day; Galilei 1898, 205).
[12] See Mach, *Mechanik*, (1902, 248, 1973, 246)
[13] Footnote to Leonardo

explanation in terms of tradition, since it only pushes the difficulty further back without resolving it (69). So why did Leonardo, Benedetti and Varron, and also later Galileo and Descartes, prefer to express the relation in terms of a proportionality to distance covered?

"The reason", writes Koyré, "seems at once both simple and profound. It is entirely a matter of the role that geometry plays in modern science, of the relative intelligibility of spatial relations" (Koyré 1978, 73). The mathematizing of the laws of nature "comes to the same thing" as the geometrization of space, "for how could something have been mathematised—before Descartes—except by geometrising it?"[14] Thus Galileo's error of using a representation that "is only valid for an increase which is uniform in relation to *time*" is attributed by Koyré to his "thorough-going geometrisation", which "*transfers to space that which is valid for time*." (78)

This is not the place to contest Koyré's grand narrative of the scientific revolution as "the geometrization of nature", which has been ably done by others.[15] But I cannot let his appeal to the notion of the "geometrization of time" go unchallenged. One finds similar Bergsonian considerations in Burtt, who contends that, as a result of Galileo's success in mathematizing motion by treating time as a straight line, "time as something lived has been banished time from our metaphysics".[16] This ignores the fact that one finds the comparison of time with a line even in Aristotle.[17] But leaving such grander claims to one side, Koyré makes two specific claims here, both of which seem doubtful, if not outright false. The first is that the mathematization of time necessarily involves its geometrization, and the second is that it is this geometrization that leads Galileo to substitute space for time in announcing his principle. Concerning the first, it is not the case that the only way to mathematize time prior to Descartes was through geometry. The idea that the successive moments of time might form an arithmetical and not a geometrical progression was far from being unavailable; indeed, as we'll see in detail in the section 4 below,

[14] "The process which gave rise to modern physics consisted in an attempt to rationalize, in other words, geometrise, space, and to mathematize the laws of nature
[15] Garber (1992); Ariew;
[16] Burtt 1924, 262; see also pp. 92-97.
[17] See Aristotle, *Physics*, 220 a8-a21 (Aristotle 1996, 107-108).

Leonardo da Vinci, Beeckman, Fabry and Cazré all characterized the succession of moments as a discrete order, and consequently represented it in terms of an arithmetical progression. Concerning the second claim, if one instead conceives time as constituting a continuous ordering—as did Galileo in asserting a 1-1 correspondence between the instants of this time and degrees of speed—it is indeed natural to represent such a continuous ordering by analogy with the ordering of points on a line.[18] One may call the employment of this spatial model to represent temporal order a "spatialization". However, this is not in itself a fallacy: there may be (indeed, are) other features of time besides the ordering of its instants, but representing a temporal ordering by analogy with a spatial ordering is traditional and unproblematic. Galileo, of course, continues to represent time in this way in his published works. The root of the fallacy we are exploring does not lie in representing time by a line—as Galileo does in the *Dialogo* (see Figure 2), where he writes of "the infinite instants that there are in the time *DA* corresponding to the infinite points on the line *DA*",[19] setting them in 1-1 correspondence with the infinite degrees of velocity of a uniformly accelerated motion—but in conceiving these same degrees of velocity to accrue uniformly in relation to the distance of the body from the beginning of its fall. If the degrees of velocity increase uniformly in time, then the degree of velocity, say, half way through the time of the fall is not the same as the degree of velocity at a point half way through the distance of the fall.

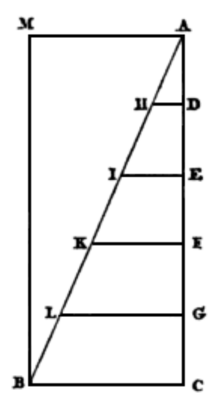


Figure 2: Galileo, *Dialogo, Second Day*

[18] Again, cf. Aristotle: "Now, what is before and after is found primarily in place. In that context it depends on position, but because it is found in magnitude, it must also be found, in an analogous fashion, in change. And since time always follows the nature of change, what is before and after applies also to time." (*Physics*, 219 a14- a18; Aristotle 1996, 105).

[19] "… gli infiniti instanti che sono nel tempo DA, corrispondenti a gli infiniti punti che sono nella linea DA" (Galilei 1897, 162).

Of course, one might argue that a more charitable reading of Koyré's point is that it is the very representation of time by a line in space that led Galileo, Descartes and others to take this false step of taking temporal distance from the beginning of the motion to be proportional to the spatial distance from it. Perhaps there is something to this. But what I wish to suggest here is that there is a much more compelling way of understanding why these authors did not see the fallacy of moving backwards and forward between talk of equal intervals of a motion through space and of equal intervals of a motion through time. This has to do with their notion of *velocity*, and its concomitant modeling by the theory of proportions.

As Enrico Giusti has argued, the operant notion of velocity in Galileo's time was the Aristotelian one: velocity (or "celerity", swiftness) is *an affection of a whole motion understood as completed*. On this conception, the greater the velocity, the less time it will take for a given body to traverse a given distance, or the greater the distance it will cover in the same time. This corresponds to what Aristotle stated in his physics:

> If one thing is faster than another, it will cover a greater distance in an equal amount of time, and it will take less time to traverse an equal distance, and it will take less time to traverse a greater distance. Some people take these properties to define "faster". (*Physics* vi.2, 232a 24-27; Aristotle 1996, 141)

Such a conception of velocity was still current in the seventeenth century,[20] as can be seen by the definition of *velox* ("fast") given by the Jesuit physicist Honoré Fabry, writing in opposition to Galileo in 1646:

[20] This notion of velocity as an affection of a body's motion taken as a whole was still current even significantly later. Giusti (1990, xxx) cites Saccheri's *Neostatica* (Milan 1708, 1), and gives the following quotation from the *De legibus gravitatis* of Paolo Frisi, who was professor of mathematics in Pisa from 1756-1764: "Celerity is that affection of a moving body which occurs so that more or less space is covered in a given time." Giusti, however, insists that Galileo employs two types of velocity in accelerated motion, the "velocity with which a moving body traverses a given line", and the degree of velocity (xxxiv). My subtle disagreement with him is that I do not think that "degree of velocity" is a velocity (even though in unpublished manuscripts Galileo does sometimes call it a velocity); here we are dealing with a stage in the transition to the concept of instantaneous velocity, but we are not there yet. See Damerow et al. (1991) for detailed arguments to the same effect.

> *Def. 2*: A fast [*velox*] motion is that by which more space is traversed in an equal time, or an equal space in less time; and a slow motion is defined contrariwise. (Honoré Fabry, *Tractatus physicus de motu locali*, Lyon 1646, p. 1).

Damerow, Freudenthal, McLaughlin and Renn have made very much the same point in their (1991), a point endorsed and aptly summarized by Vincent Jullien and André Charrak in their study of Descartes' writings on free fall as follows: "Speed, such as it is in usage in the pre-classical tradition (before Galileo), which P. Souffrin has called 'holistic speed' ('global speed' might be preferable), is the measure of a movement accomplished in an elapsed time and/or a space traversed." (Damerov et al. 2002, 37-38)[21]. Giusti (1990, xxxiv) calls this the *velocità complessive*, 'overall velocity', which is the term I shall use.

Now it is crucially important to realize that this is not just a term, but a concomitant of how motion was represented mathematically. Velocities are affections of motions, and these are compared quantitatively using proportions. It is true that Aristotle did not compare velocities directly, but as Clagett explains (1959, 217), the philosophers of the Merton School interpreted his definition to allow this. Thus, according to Thomas Bradwardine, "for every two local motions continued through the same or equal times, the velocities and spaces are proportional, so that one of the velocities is to the other as the space traversed by one velocity is to the space traversed by the other", and "for every two local motions over the same or equal spaces, the velocities and times are always inversely proportional, so that one of the velocities is to the other as the space traversed by one velocity is to the space traversed by the other" (Clagett 1959, 233). That is, if $T_1 = T_2$, then $V_1:V_2 = S_1:S_2$, and if $S_1 = S_2$, then $V_1:V_2 = T_2:T_1$. But these are the velocities with which those motions are accomplished, overall velocities, not velocities at a time, as in a contemporary understanding. Again, this concurs with Jullien and Charrak, in their summary of the

[21] As noted earlier, on discovering Jullien's and Charrak's book in the late stages of composition of this chapter, I discovered that many of my points about the anachronism of reading the modern functional view back into the originators of classical mechanics had already been made with great clarity in Damerow et al (1991). As Jullien and Charrak remark, summarizing Damerow et al: "Elle consiste en un vaste anachronism en vertue duquel on mobilise les concepts «classiques» de vitesse instantanée, de summation intégrale (ou au moins de limite) et de function… " (2002, 36-37).

conclusions of Damerov et al.: "The terms speed, *velocitas* or *celeritas*, employed by themselves, do not designate the speed in an instant, or in 'a point in the trajectory'." (2002, 37)

This contrasts with the reasonings of Mach, Tannery and Koyré expounded above, who read back into the work of Galileo and his predecessors the conception that Newton's work has accustomed us to, where there is a velocity of motion at every single instant. To Galileo and his contemporaries, however, such a notion of instantaneous velocity would have appeared self-contradictory. There cannot be any motion in an instant, since a motion must take place over time. So, if velocity is an affection of a motion, there cannot be such a thing as an instantaneous velocity.

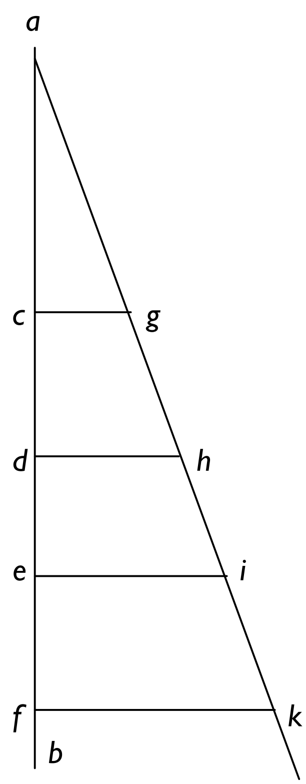
This, of course, is why Galileo (following in the tradition of the Merton School and Oresme) adopted and developed the notions of *degree of velocity* and *moment of velocity*: precisely in an attempt to explicate how an accelerated motion gets progressively faster without assuming a motion in an instant. He does not have the modern concept of an integral of instantaneous velocities with respect to time[22]; instead he appropriates the idea initiated by the Merton school that a motion has a certain *intensity* at any given instant. Each such intensity can be represented quantitatively by a transverse line, in such a way, Galileo asserts, that "all the degrees of velocity" can compose into the overall velocity of the motion in the same way that "all the lines" can be seen to add up to the whole corresponding area. Thus in Folio 128 of the *Fragments connected with the Discorsi* Galileo wrote:

> Thus the degrees of velocity continually increase at all the points of the line *af* according to the increment of the parallels drawn from all these same points. Moreover, because the velocity with which the moving body has come from *a* to *d* is composed of all the degrees of velocity acquired in all the points of the line *ad*, and the velocity with which it has traversed the line *ac* is composed of all the degrees of velocity that is has acquired in all the points of the line *ac*, it

[22] Again, much the same point has been lucidly argued by Damerow, Freudenthal, McLaughlin and Renn, who write: "From the point of view of Aristotelian natural philosophy—where no equivalent to a functional dependence of motion on a certain parameter exists, and where the velocity of motion always refers to its overall extension in space and time—the alternative: velocity stands either in such a specified relation to space or else to time, cannot sensibly be posed." (1991, 1-2).

follows that the velocity with which it has traversed the line *ad* has the same proportion to the velocity with which it has traversed *ac*, as all the parallel lines drawn from all the points of the line *ad* up to *ah* have to all the parallels drawn from all the points of the line *ac* up to *ag*.[23]

Figure 3: Galileo, *Frammenti attententi ai Discorsi*



Here we see that the conception is that all the degrees of velocity add up to a velocity, but this velocity is the overall velocity, Giusti's *velocità complessive*, the swiftness with which the motion is accomplished. By this construction Galileo has proved this to be proportional to the area of the corresponding triangle, and thus to the square of the distance of fall. But now he concludes:

[23] "Vanno dunque continuamente crescendo i gradi di velocità in tutti i punti della linea *af* secondo l'incremento delle parallele tirate da tutti i medesimi punti. In oltre, perché la velocità con la quale il mobile è venuto da *a* in *d* è composta di tutti i gradi di velocità auti in tutti i punti della linea *ad*, e la velocità con che ha passata la linea *ac* è composta di tutti i gradi di velocità che ha auti in tutti i punti della linea *ac*, adunque la velocità con che ha passata la linea *ad*, alla velocità con che ha passata la linea *ac*, ha quella proporzione che hanno tutte le linee parallel tirate da tutti i punti della linea *ad* sino alla *ah*, a tutte le parallele tirate da tutti i punti della linea *ac* sino alla *ag*." (*Opere*, VIII, p. 373)

> Thus the velocity with which the line *ad* is traversed to the velocity with which the line *ac* is traversed has double proportion to that between *da* and *ca.* And since the ratio between the velocities is the inverse of the ratio of the times (for to increase the velocity is the same as to decrease the time) it follows that the time of the motion through *ad* is to the time of the motion through *ac* as the subduplicate proportion of the distance *ad* to the distance *ac*. Thus the distances from the point of departure are as the squares of the times, and, dividing, the spaces traversed in equal times are as the odd numbers from unity… .[24]

The wording here provides confirmation that Galileo's concept of velocity in 1604 is the Aristotelian one: it is "the time of the motion through a given line", not the instantaneous velocity at the end of the motion. In keeping with this concept he reasons that the times are inversely as the overall velocities, so that, since the velocities are as the square roots of the distances, the times are as the square roots of the distances, and "the distances from the point of departure are as the squares of the times". This is a mistake, since what should follow is that the times are *inversely* as the square roots of the distances. This can be seen more clearly using (modern) proportions: $t_{ad} : t_{ac} = v_{ac} : v_{ad} = \sqrt{ac}:\sqrt{ad}$. $\therefore\ ac : ad = t_{ad}^2 : t_{ac}^2$. But Galileo does not notice this error, seduced, no doubt, by the obviousness of the fact that as the body falls, its distance and time from the starting point both continually increase.[25]

Subsequently, though, Galileo did come to realize that there is an incompatibility between his t^2 law and the principle on which he had based it, that the velocity of the falling body is proportional to its distance from the beginning of its motion. Guiding him in finding the right principle, as he explains in the

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[24] "Adunque la velocità con che si è passata la linea *ad*, alla la velocità con che si è passata la linea *ac*, ha doppia proporzione di quell ache ha *da* a *ca*. E perché la velocità alla velocità ha contraria proporzione di quella che ha il tempo al tempo (imperò che il medesimo è crescere la velocità che sciemare *a* tempo), adunque il tempo del moto in *ad* al tempo del moto in *ac* ha subduplicata proporzione di quella che ha la distanza *ad* alla distanza *ac*. Le distanze dunque dal principio del moto sono come i quadrati de i tempi, e, dividendo, gli spazi passati in tempi eguali sono come i numeri impari ab unitate… " (Galilei 1898: 373-4)

[25] My analysis here agrees with that of Giusti, who criticizes Galileo for an equivocation in his use of the term "inverse proportion (*contraria proporzione*)" (Giusti 1990, xxxvi); but it is at odds with that of Jürgen Renn in (Damerow et al, 1992). Renn claims that Galileo's reasoning has "the advantage [over Descartes'] of yielding the correct result". For, unlike Descartes, Galileo "inverts … the double proportionality of the velocities by a transition to a 'half' or 'mean' proportionality of the times thus obtaining the law of fall in its mean proportional form" (167). I do not see that this re-expressing of the proportion also involves an inversion.

Discorsi, was above all the Aristotelian notion of "the intimate relationship between time and motion"; this, together with the idea that "we find no addition or increment simpler than that which repeats itself always in the same manner",[26] he was led to the correct principle that a motion would be uniformly and continuously accelerated "when, during any equal intervals of time whatever, equal increments of velocity are given to it"[27]. The degrees of velocity, that is, must be conceived as increasing at successive instants of the time of fall, and not at successive distances from the beginning of the motion.

The subtlety of these considerations and awareness of his previous error accounts for the circumspection with which Galileo treats the topic of uniformly accelerated motion in the *Discorsi*. He is careful not to talk of the velocities of a body moving with uniform acceleration, but the degrees of velocity (*velocitatis gradus*) it has at each of the instants of its motion, thus staying close to the medieval tradition. His famous so-called Mean Speed Theorem, Theorem 1 of his account of accelerated motion on the Third Day of the *Discorsi*, does not refer to the mean as one between "the highest speed and the speed just before acceleration began", as Crew and de Salvio translate it (1914, 173). The comparison is between a body in uniformly accelerated motion and the same body moving uniformly with a *degree of velocity* equal to one half of the final degree of velocity of the accelerated one:

> The time in which a given space is traversed by a moving body with a motion uniformly accelerated from rest is equal to the time in which the same space would be traversed by the same body moving with an equable motion whose degree of velocity is one half of the last and greatest degree of velocity of the preceding accelerated motion.[28]

[26] "Ora, se consideriamo attentamente la cosa, non troveremo nessun aumento o incremento più semplice di quello che aumenta sempre nel medesimo modo. Il che facilmente intenderemo considerando la stretta connessione tra tempo e moto…" (Galilei 1898: 197)
[27] "[lo possiamo] in quanto stabiliamo in astratto che risulti uniformemente e, nel medesimo modo, continuamente accelerato, quel moto che in tempi eguali, comunque presi, acquista eguali aumenti di velocità." (Galilei 1890–1909, 8:197–98).
[28] "Il tempo in cui uno spazio dato è percorso da un mobile con moto uniformemente accelerato a partire dalla quiete, è eguale al tempo in cui quel medesimo spazio sarebbe percorso dal medesimo mobile mosso di moto equabile, il cui grado di velocità sia sudduplo [*la metà*] del grado di velocità ultimo e massimo [raggiunto dal mobile] nel precedente moto uniformemente accelerato "(Galilei, 1898, 208)

This repeats the same argument Galileo had given earlier in the *Dialogo*: there a body moving with an equable motion whose constant degree of velocity is equal to "the greatest degree of velocity acquired by the moving body in the accelerated motion" will cover twice the space in the same time.[29] This should be compared to the analogous application of the Merton Rule to uniformly decelerated motion by Nicolas Oresme:

> If *a* is moved uniformly for an hour and *b* is uniformly decelerated in the same hour from a degree [of velocity] twice [that of *a*] and terminating at no degree, then they will traverse equal distances, as can easily be proved. Therefore, by the definition of velocity, it ought to be conceded that they were moved equally quickly for the whole hour. Therefore, the whole motion of *b* ought not to be said to be as fast as the maximum degree of velocity.[30]

Here Damerow and Freudenthal make some cautions, following Anneliese Maier (Damerow et al. 1991, 18-19). In Oresme, and in the later representations, this is simply a method of graphical representation which shows how the variation of qualities can be depicted, rather than a method of calculation. Something similar seems to be implied in Galileo's usage: if the area represents the velocity, the lines represent the degrees. There is no question of "integrating" infinitesimally thin lines to make a surface.

With these considerations in mind, let us examine a typical argument of one of Galileo's predecessors, before we proceed to his criticisms. In his *De motu tractatus* of 1584, Michel Varron wrote:

> The spaces of this motion conserve this proportion of celerity, so that whatever the ratio of the whole space through which the motion is made to the part, (the beginning on both sides is assumed to be where the beginning of the motion is), the ratio of celerity to celerity is the same. For example, if some force will move through the line ABE, and AB is part of this line, then the

[29] "E sì come la BC era massima delle infinite del triangolo, rappresentanteci il massimo grado di velocità acquistato dal mobile nel moto accelerato … passi con moto equabile nel medesimo tempo spazio doppio al passato dal moto accelerato" (Galilei 1897, 163; 1953, 229)

[30] Oresme 1968, 559-561; quoted from Damerow et al. 1991, 18.

> ratio of AE to AB will be the same as that of the celerity of the motion at the point E to the celerity at the point B.
>
> A proportion of this kind is observed in the parallels cutting a triangle… So if the space is divided into aliquot parts, at the end of the second space it will be carried twice as fast as at the end of the first…[31]

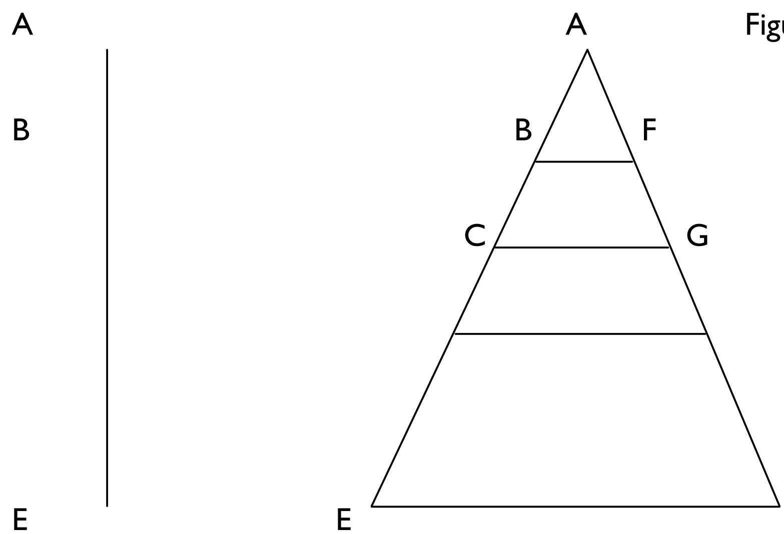

Figure 4: Michel Varron, 1584

There is ambiguity here: Varron talks of the body being carried twice as fast at the end of the second space, which certainly looks like an instantaneous velocity. Nevertheless, he is (like Galileo) applying the theory of proportions: if it is carried twice as fast through the second space of its motion, it will complete that space in half the time. This requires the celerity to be taken in the Aristotelian sense as the swiftness of the motion taken as a whole.

Now let us turn to the way in which Galileo criticizes his predecessors. The passage in question runs as follows:

> When the velocities have the same proportion to the spaces traversed, or to be traversed, such spaces come to be traversed in equal times; for if the [velocity][32] with which the falling body

[31] Michel Varron, *De motu tractatus*, Geneva, Jacob Stoer, 1584, p. 12ff.

> traverses the space of four cubits be twice the [velocity] with which it traverses the first two cubits (seeing as the former space is double the latter), then the times of such traversals will be equal; but for the same mobile to traverse the four cubits and the two in the same time could take place only in an instantaneous motion; but we have seen that the heavy body makes its falling motion in time, and traverses the two cubits in a smaller time than the four; therefore it is false that the velocity increases as the space.[33]

Alexandre Koyré calls this a "specious argument" and a "thoroughly mistaken" refutation of his predecessors' views (78). Again citing Mach and Tannery, he writes: "The argument contains a similar error to that which we found in the argument discussed above: Galileo applies to motion of which the increase of speed is proportional to the distance covered a calculation which is only applicable to uniformly (in relation to time) accelerated motion."[34]

We may explicate Koyré's criticism as follows. Galileo applies his Mean Speed Theorem (valid only for motion uniformly accelerating with time), to obtain $s = {}^1/_2 vt$, where v is the final velocity. This yields $t = 2s/v$. So if t_1 and t_2 are the times of fall through 2 and 4 cubits resp., $t_1 = 4/v_1$, and $t_2 = 8/v_2$. But since $v_2 = 2v_1$, $t_2 = 8/2v_1 = 4/v_1$, and we have $t_2 = t_1$, and the time of fall through the second two cubits is zero. So, according to Koyré, Galileo has generated this contradictory conclusion by applying to the case of a velocity increasing with distance a calculation valid only for a velocity increasing with time.

But I think much more sense can be made of Galileo's refutation as follows. Even though it is very tempting to read Varron, as did Mach, to intend by his expression "the celerity of motion at E" the

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[32] Here I am reading the singular 'la velocità' and 'della velocità' for the text's plurals, 'le velocità' and 'delle velocità', which do not appear to make sense in the context, although it is perhaps explicable in terms of an implicit comparison of pairs of velocities in a proportion.

[33] "Quando le velocità hanno la medesima proporzione che gli spazii passati o da passarsi, tali spazii vengon passati in tempi eguali; se dunque le velocità con le quali il cadente passò lo spazio di quattro braccia, furon doppie delle velocità con le quali passò le due prime braccia (sì come lo spazio è doppio dello spazio), adunque i tempi di tali passaggi sono eguali: ma passare il medesimo mobile le quattro braccia e le due nell'istesso tempo, non può aver luogo fuor che nel moto instantaneo: ma noi veggiamo che il grave cadente fa suo moto in tempo, ed in minore passa le due braccia che le quattro; adunque è falso che la velocità sua cresca come lo spazio." (Galileo 1898, 203-204)

[34] Koyré 1978, p. 116, fn. 51. Koyré cites in this connection Mach's *Mechanik* (Mach [1883] 1973, 245), and also Paul Tannery, *Memoires Scientifiques*, vol. VI, p. 400 ff.

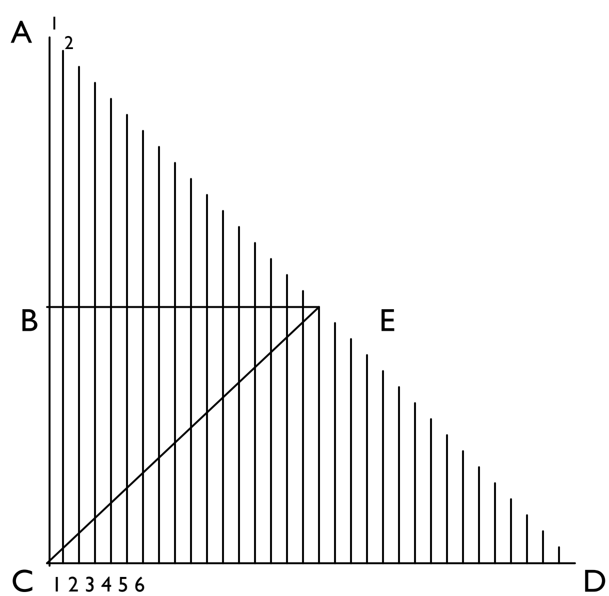
instantaneous velocity at that point, there is no independent evidence to suggest that Varron has made this conceptual advance—one that, I stress, Galileo himself did not make. It is not obvious that Varron is using the word *celeritas* as a synonym for intensity of motion or degree of speed, so even if there is something of that in his conception, Galileo seems to be interpreting *celeritas* as a synonym for *velocitas*, the swiftness of the whole motion calculated as terminating at E.

So, I contend, Galileo is not misapplying his Mean Speed Theorem. He is interpreting his predecessors as holding that the overall *velocitas* of one motion to another is proportional to the respective spaces traversed in the same times, as he himself had previously done. The more quickly a given space is covered, or the more space that is completed in a given time, the swifter the motion. According to this concept, a motion twice as swift will cover twice the distance in the same time. So if the first two cubits are traversed with an overall velocity v_1, and the whole four cubits with a velocity $v_2 = 2v_1$, then the times of fall will be $t_1 = 2/v_1$, and $t_2 = 4/v_2 = 2/v_1$, and we will have $t_2 = t_1$, so that the time of fall through the second two cubits is 0, as Galileo argued. If this is the correct interpretation of Varron's argument, then Mach's analysis is in error in imputing to him the modern concept of instantaneous velocity. Moreover, on this interpretation Galileo's *reductio* goes through, and no paralogism is involved. There may well be a rhetorical component to his argument, however, if it is the case that Varron is confusing the intensity of the motion at a given instant (Galileo's "degree of velocity") with this concept of overall *velocitas*. In that case, Galileo, having successfully disentangled them, is attributing only the latter to his predecessors in order to make his point.

3. DESCARTES' MODEL OF FREE FALL

Now let us turn our attention to Descartes. The French savant had first proposed a solution to the problem of a heavy body falling in a vacuum under the influence of a constant force of attraction in response to a very neat formulation of the problem by his Dutch mentor Isaac Beeckman in 1618. I have analyzed that episode elsewhere (see Arthur 2007, Arthur 2011). What concerns me here is Descartes' abiding understanding of what he had thought himself to have proved then, as evidenced in his letters to Mersenne of 13th November 1629 and 14th August 1634 (pp. 9 and 44 in CSM-K).[35] In the first of these he presents a diagram in which AC represents the distance through which the body falls, and also represents the motion of a body which, having received an impulse at A, travels at a uniform speed from A to C for the whole duration. This is in keeping with what Descartes had learned from Beeckman, that “in a vacuum, what has once begun to move keeps on moving at the same speed”, and that the action of gravity can be analyzed in terms of impulses given to the falling body at successive moments of the fall. Thus the vertical lines 2, 3, 4, etc. represent the distances that would be traversed by the same body if it moved solely by virtue of the additional impetuses, each equal to the first, received in the second, third, etc. moments of its fall. (Each is shorter than the last, in that the body, travelling at the same speed, would have a slightly shorter time of fall.) Accordingly, the force of the body’s motion in the first moment will be proportional to AC; in the second moment this force is maintained, and added to it is a second force represented by the line 2, and so on. The horizontal lines (such as BE) would then represent the force or intensity of motion at each subsequent moment. Says Descartes,

Figure 5: Descartes to Mersenne, 1629



[35] Since writing this section I have discovered the detailed analyses of Descartes’ mathematical treatments of the law of fall in Damerow and Freudenthal (1991, chapter 2), and Jullien and Charrak (2002). But my analysis of the “4/3 proportion” differs from theirs, as we shall see.

> Thus we get the triangle ACD, which represents the increase in the velocity of the weight as it falls from A to C, the triangle ABE representing the increase in the velocity over the first half of the distance covered, and the trapezium BCDE representing the increase in the velocity over the second half of the distance covered, namely BC. Since the trapezium is three times the size of triangle ABE (as is obvious), it follows that the velocity of the weight as it falls from B to C is three times as great as what it was from A to B. If, for example, it takes 3 seconds to fall from A to B, it will take 1 second to fall from B to C. Again, in 4 seconds it will cover twice the distance it covers in 3, and hence in 12 seconds it will cover twice the distance it does in 9, and in 16 seconds four times the distance it covers in 9, and so on in due order.

Commenting on this passage, the editor of the standard English translation of Descartes' writings (for this section, Dugald Murdoch) says "Descartes wrongly takes the line ABC to represent the time, instead of the distance travelled; this leads him to take the distance travelled as being proportional, not to the square of the time ($d = {}^1/_2 gt^2$), but to a power of the time, the exponent of which is $\log 2/\log {}^1/_3$." (Descartes 1991, 9, n. 1).

Murdoch does not give the calculation. But here is a reconstruction. Descartes is assuming that the velocity of the falling body is proportional to the distance traversed, and inversely proportional to the time elapsed. Assuming velocity means the instantaneous velocity, and that it is expressible as a function of time elapsed and of distance covered at a given time, we have

$$v = \mathrm{d}s/\mathrm{d}t = \mathrm{k}s/t$$

The solution to this is $s = \mathrm{c}t^{\mathrm{k}}$, where s is the distance fallen, t the time, and c and k are constants. The first two examples Descartes gives say that twice the distance is done in 4/3 of the time, and the third example that $4s : s$ is as 16t : 9t. These relations fit the formula $s_1 : s_2 = t_1^{\mathrm{k}} : t_2^{\mathrm{k}}$, and yield $2 = (4/3)^{\mathrm{k}}$ or $\mathrm{k} = \log 2/\log {}^4/_3 = \log 2/(\log 4 - \log 3)$. (Here the logarithms may be taken to any base; Murdoch's $\log {}^1/_3$ instead of $\log{}^4/_3$ (= 2 $\log{}^1/_3$) in the denominator appears to be an error.) In a precisely similar calculation, Damerow and Freudenthal (1991, 59) arrive at the formula[36]

$$s = \mathrm{c}\, t^{\log 2/(\log 4 - \log 3)}$$

Neither Damerow and Freudenthal nor Murdoch reference Paul Tannery's note in his and Adam's edition of Descartes' works, although this is the probable source, at least of Murdoch's own note. There Tannery remarks that Descartes "therefore comes to a relation essentially different from that of Galileo, since it would amount to considering the space traversed as proportional, not to the square of the time, but to a power of time whose exponent is the ratio of log 2 to log 4/3, that is to say, about 2.4."[37] Jullien and Charrak do reference this note, rightly calling it "a little clumsy, because anachronistic" of Tannery to "construct an exponential function giving the spaces in relation to the variable *time*".[38]

It certainly is anachronistic. Given that Descartes eschewed transcendental curves from his geometry, let alone that he knew nothing of functions or the integral calculus, this attribution to him of an $s = \mathrm{c}t^{\mathrm{k}}$ law is unconvincing, to say the least.[39] Moreover, it again attributes to Descartes a conception of instantaneous velocity. But inspection of these passages shows Descartes writing of "the velocity of the weight as it falls from B to C". This is nonsense if one has a conception of velocity as constantly increasing as the body falls, and is only interpretable, I submit, if velocity here, as above, is an affection of the whole motion, the swiftness with which this whole (leg of the) motion is accomplished.

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[36] I have no explanation of why, having correctly identified the Aristotelian theory of proportions at work in Descartes' mathematization, these authors go on to give an explanation of Descartes' reasoning that appeals to the functional interpretation they earlier decry.

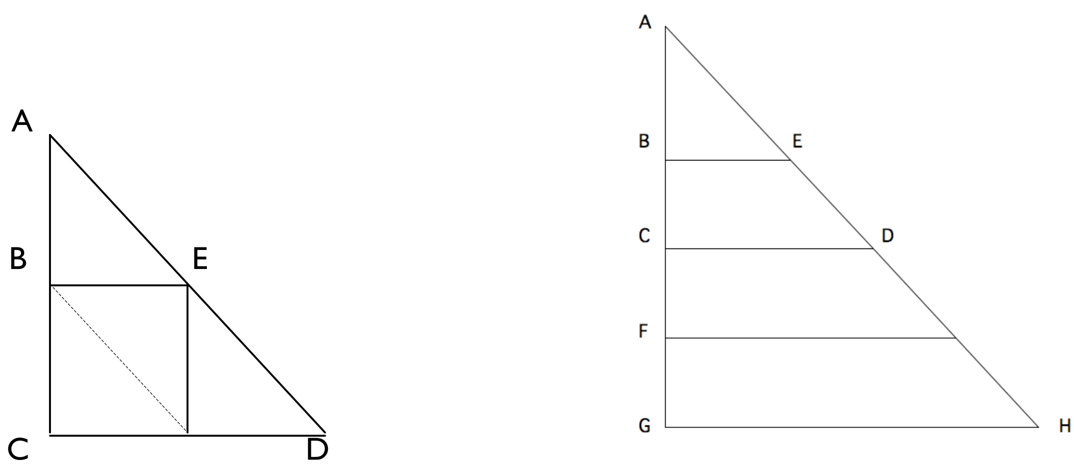
[37] "Il aboutit donc à une relation essentiellement différent de celle de Galilée, puisqu'elle reviendrait à considérer l'espace parcouru comme proportionnel, non pas au carré du temps, mais à une puissance du temps dont l'exposant est la rapport de log.2 à log4/3, c'est à dire environ 2,4." (AT I 75). As Jullien

[38] "C'est le sens d'une note maladroit, car anachonique, de Tannery, qui construit une function exponentielle donnant les espaces par rapport à la variable *temps*." (Jullien and Charrak 2002, 121).

[39] In a letter to Mersenne in October 1631 Descartes writes that if a void and constant action of gravity are accepted, "there would be no means of explaining the speed of this movement by numbers other than the ones I have sent you, at least ones that are rational; and I do not even see that it would be easy to find irrational ones, nor any line in geometry which would explicate them better". Tannery apparently took Descartes' remark to licence an interpretation in terms of exponential powers, but as Jullien and Charrak argue (1992, 123), it is more natural to see him as rejecting any such an interpretation as unphysical.

Accordingly, my interpretation is as follows: a motion that takes three seconds to cover AB is three times less fast (*velox*) than one that takes one second to cover the equal distance BC. Thus if the *velocitas* for AB is $^{1}/_{3}$ of that for BC —since ABE = $^{1}/_{3}$ BCDE— and it covers AB in 3 seconds, it will cover BC in 1, and AC in 4. If it covers AB in 9 seconds, it will cover BC in 3, and thus AC in 12. And if it covers AB in 9 seconds, and thus AC in 12, since ACD:CDHG is also 1:3, it will cover CG in 4 seconds, and thus ACG in 16.[40]

Figure 6: Descartes to Mersenne, 1634



Thus if the velocity through the space AB, represented by ABE, is 1 unit, that through BC is 3 units, that through CF is 5 units, that through FG is 7 units. This is the odd number rule that Descartes has discovered. It is not identical to Galileo's rule, contrary to Descartes' impression, where it is the velocity over each equal part of the uniformly accelerated motion, reckoned time-wise, that increases as the odd numbers. The reason why Descartes does not see the discrepancy is that he has reasoned as follows

> If successive equal spaces are covered by the moving body in times of ratios 1, $^{1}/_{3}$, $^{1}/_{5}$, $^{1}/_{7}$, … to 1, then in successive equal times the spaces covered are of ratios 1, 3, 5, 7, … to 1.

[40] Here the first diagram is from the 1634 letter, with the lettering of the 1629 letter. The second is my own.

In other words, he has applied the rule:

> A body travelling with N times the velocity will traverse an equal space in $^1/_N$ of the time; or in an equal time will traverse a space N times as great.

But this is just to apply the Aristotelian conception of *velocitas*, the overall velocity discussed above to successive portions of the motion.

It is, I submit, not surprising that Descartes does not see the discrepancy between his results and Galileo's. For in the case of equal times, they do indeed get identical results. If the individual spaces covered in equal times are in the ratios 1, 3, 5, 7, … to 1, then the overall spaces covered are as the squares of the times.

What is wrong with Descartes' conceptual apparatus can be seen by looking at the motion through four equal spaces, say four cubits, in two *spatial* halves. By hypothesis, the moving body covers these in times of ratios 1, $^1/_3$, $^1/_5$, $^1/_7$, to 1. Thus the overall times of fall through 1, 2, 3, and 4 cubits are 1, $^4/_3$, $^{23}/_{15}$, $^{176}/_{105}$, resp. Now looking at this in two halves, the first 2 cubits are covered in $^4/_3$ s, the second 2 cubits in $^{176}/_{105} - {}^4/_3 = {}^{36}/_{105} = {}^{12}/_{35}$ s. Thus the times of these two equal halves of the motion are in the ratio $^4/_3 : {}^{12}/_{35}$ = 35: 9, i.e. almost 4 : 1, whereas the velocities are in the ratio 4:12, or 1:3.

In sum, Descartes gets the right result by a compensation of errors, as Koyré pointed out. But this compensation consists in his using the Aristotelian law of *velocitas* to transpose the right result—obtained correctly in 1618 as a result of Beeckman's having set the problem up in terms of times—into a result in terms of equal spaces, and then back again. It is not a misreading of a proportionality between instantaneous velocity and time elapsed as equivalent to a proportion between instantaneous velocity and space covered. Neither Descartes, nor Galileo, nor Varron, made that particular error. For none of them *had* the concept of instantaneous velocity. And none of these historical episodes can be understood using the terms of this modern conceptual apparatus without hopeless distortion.

4. CLAGETT AND THE MERTON SCHOOL

Now, at this point I anticipate that it might be objected that I have exaggerated the remoteness of seventeenth century conceptions of motion from ours to such a degree as to obscure the continuity of Galileo's conceptions with the modern understanding. For although Descartes apparently never recognized the error involved in his analysis of the problem of fall, Galileo did correct his analogous error. And his concept of the *degree of velocity* is, even if not technically a *velocity* for the reasons given above, clearly the source for Newton's concept. Implicit in Galileo's concept of degree of velocity, as in the young Newton's idea of a velocity at an instant (what he later called a *fluxion*), is the idea that this is the velocity that a body would move at if it were to continue moving uniformly with the same degree of velocity for longer than an instant. Moreover, as Clagett has observed, this same concept can be found in the earlier Merton School of fourteenth century Oxford. In a passage from a fragment *On Motion* attributed to Richard Swineshead, for example, we find:

> The reason why the velocity of this motion will be attended by a described line belonging to it is this: to each degree [of velocity] in a local motion there corresponds a certain lineal distance which would be described in some time by a motion with just such a degree [of velocity] throughout.[41]

His contemporary William Heytesbury proposed a similar conception in the following passage describing nonuniform motion:

> In nonuniform motion, however, the velocity will be attended at each instant by a line belonging to it which would be described by a point moving with the fastest motion if it were moved

[41] "Causa autem quare penes lineam descriptam velocitas illius motus attendit, est hoc: cuicunque gradui in motu locali correspondet certa distantia linealis quae in tanto tempore et in tanto cum partibus tali gradu describeretur." (Clagett 1959, 245)

> uniformly for a time with that degree of velocity by which it is moved in the same instant, at any given instant whatever.[42]

Clagett translates the beginning of this passage as "In nonuniform motion, however, the velocity at any given instant will be measured by the path which would be described by the most rapidly moving point …" (236).[43] Consequently, he attributes to Heytesbury the concept of velocity at an instant, or *instantaneous velocity*: "For him instantaneous velocity is to be measured or determined by the path which *would* be described by a point if that point were to move during some time interval with a uniform motion of the velocity possessed at the instant." (237) If, however, we bear in mind the concept of velocity I have been urging as the norm, the velocity in question is that of the whole motion: what Heytesbury and Swineshead are both doing in these passages is justifying their representation of the intension of motion at any instant by an extended line representing the degree of velocity at that instant.

Clagett draws attention to Galileo's employment of the terminology and conceptual apparatus of the Merton School (237), giving an excerpt from the *Discorsi* in Crew and de Salvio's translation, which he has altered in parts to a "very literal" translation so as "to reveal more clearly Galileo's dependence on the Merton vocabulary" (251):

> To put the matter more clearly, if a moving body were to continue its motion with the same degree or moment of velocity (*gradus seu momentum velocitatis*) it acquired in the first time-interval, and continue to move uniformly with that degree of velocity, then its motion would be twice a slow as that which it would have if its velocity (*gradus celeritatis*) had been acquired in two time intervals. And thus, it seems, we shall not be far wrong if we assume that increase in velocity (*intentio velocitatis*) is proportional (*fieri juxta*) to the increase of time (*temporis extensio*).

[42] "In motu autem difformi, in quocunque instanti attendetur velocitas penes lineam quam describeret punctus velocissime motus, si per tempus moveretur uniformiter illo gradu velocitatis quo movetur in eodem instanti, quocunque dato." (Clagett 1959, 240)
[43] The expression "by the most rapidly moving point" occurs because these authors are considering the motion of a rotating radius. See Clagett 1959, 216.

This is all very revealing. Clagett has correctly substituted "degree or moment of velocity" for Crew and de Salvio's plain "speed", but he has left unchanged their translation of *gradus celeritatis* as "velocity" when a literal translation of the Latin would be "degree of swiftness". He has then changed their "increment of speed … proportional to increment of time" to "increase in velocity … proportional to increase of time", when a "very literal" translation would be "if we suppose the intension of velocity to occur in proportion to the extension of time". Here Galileo is following the Merton theorists in representing the intension or degree of velocity at each instant by a line proportional to the time of fall—"the extension of time". Clagett, on the other hand, concludes that "Galileo in this passage compared the instantaneous velocities at the end of the first time-period and at the end of the second time-period." As should by now be clear, I believe that this is an anachronistic projection of a modern conceptual understanding back onto Galileo and the Merton School. It echoes, and perhaps has its source in, Koyré's comment on this same passage: "'The intension of the speed' or 'the degree of speed' is the instantaneous speed of the moving body." (Koyré 1978, 124, n. 136). For further discussion of the gulf between the medieval and modern representations of motion, I defer to the excellent analysis given by Damerow et al. 1992.

5. MERSENNE AND DISCRETIST ACCOUNTS OF FREE FALL

Again, I am not denying that Galileo's ideas are the proximate source of our modern conception. But the transformation of his ideas was a complex historical process involving many aspects that I can at best gesture at here. The most salient consideration is that we have so far been treating motion entirely in isolation from any consideration of its cause. For even if Galileo's mathematical treatment of nonuniform motion is accepted, in the absence of an account of the cause of this motion one is not obliged to agree that this is how free fall actually occurs. Now, if there is one thing that the proponents of the emerging Mechanical Philosophy did not dispute in Aristotle's physics, it was that all changes of motion are effected by bodies in contact. Given this, whatever the precise cause of gravity, it seemed to most seventeenth century thinkers that the action on a body resulting in its falling at an accelerated rate should be explained in terms of the impacts on it of other moving particles: hence the heavy emphasis on the rules of collision. Thus one could accept Galileo's principle that increases in velocity are always proportional to the time of fall without having to accept his claim that in actual fact acceleration does "occur continually from moment to moment, and not interruptedly from one quantified part of time to another".[44]

This consideration was particularly germane to the reception of and development of Galileo's account of fall. At the center of the dissemination of Galileo's work in France and beyond was the Minim Father Marin Mersenne, who discussed in several works of the 1630s and 40s. As has been demonstrated with great lucidity by Carla Rita Palmerino (1999, 2010), Mersenne's growing skepticism over the truth of Galileo's account of fall during this period was largely conditioned by such worries. In particular, there were two difficulties raised by Descartes that had ever greater influence on Mersenne's skepticism regarding Galileo's account: his conviction that there had to be an initial impact, and that a body could not in fact go through an actual infinity of degrees of speed.[45] These two worries combined to suggest that a more acceptable account of fall might still be consistent with the empirical facts: a very large but finite number of discrete increases in uniform motion caused by the impacts of subtle matter, each lasting for a very small but finite moment.

As I have argued elsewhere (Arthur 2011), precisely such a model had been suggested by Beeckman to Descartes in their first encounter in 1618. Beeckman had conceived the acceleration of the falling body to occur by successive discrete tugs resulting in successive equal increments of the force of motion in equal moments, with the previously acquired increments in the force of motion being conserved. Thus in the second moment it has twice the force of motion as in the first, and so covers twice the space it did in that moment. So as a result of these discrete tugs occurring at the beginning of each successive equal moment or physical indivisible of time, the speeds will be as 1, 2, 3, 4, etc. in the successive moments;

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[44] "Ma perchè l'accelerazione si fa continuamente di momento in momento, e non intercisamente di parte quanta di tempo in parte quanta…" (*Dialogo*; Galilei 1897, 162).
[45] See Palmerino 2010 for details.

and given the equality of moments, the distances will therefore also be as 1, 2, 3, 4, etc. in the successive moments, i.e. in arithmetical progression.[46] Thus in two hours each divided into 4 moments, the speeds will be as 1, 2, 3, 4, and 5, 6, 7, 8, with proportional distances covered in each moment. So after one hour the total distance covered would be as 1 + 2 + 3 + 4 = 10, and after 2 hours, 10 + 5 + 6 + 7 + 8 = 36, giving the ratio of the space covered in one hour to that covered in two as 10:36. By increasing the number of moments in each hour, the ratio can be made arbitrarily close to the ratio 1:4, as Beeckman noted in his diary in 1618. One can, in fact, derive a general formula. If n is the number of moments into which each hour is divided, the ratio between the distance D_1 covered in the first hour to the distance D_2 covered in two hours will be

$$D_1{:}D_2 \;=\; {}^1/_2\, n/(n+1) : n\,(2n+1) = \;(1 + {}^1/_n) : (4 + {}^2/_n)$$

Almost thirty years later, as Palmerino (2010, 59-61) reports, Fabry made exactly the same argument in his *Tractatus physicus de motu locali*:

> *A naturally accelerated motion is not propagated through all degrees of slowness.* For there are as many degrees of this propagation as there are instants through which this motion endures, since in every single instant a new accession of impetus occurs; but there are not infinitely many instants, as we will demonstrate in our *Metaphysics*."[47]

Thus each actual moment or instant corresponds to the duration of a uniform motion, with successive increases in this motion caused by the collision or tug of whatever particles are responsible for gravity. Consequently, there was no way to decide empirically between Galileo's continuist law and this discretist rival: the moments could always be supposed small enough to make any discrepancy smaller than what could be empirically discerned. But the discretist law of Beeckman and Fabry had the dual advantage that

[46] See Arthur 2011 for details.

[47] "*Motus naturaliter accelerato non propagatur per omnes tarditatis gradus;* quia tot sunt huius propagationis gradus, quot sunt instantia, quibus durat hic motus, cum singulis instantibus nova fiat impetus accessio, sed non sunt infinita instantia, ut demonstrabimus in Metaphysica…" (1646, 96)

it did not presuppose an infinity of degrees of speed, and that it conformed to the kind of causes sanctioned by the mechanical philosophy.

6. CONCLUSION

In conclusion, let me now turn to the historiographical issues I raised in section 1. What I have tried to show in adequate detail is that the projection back onto Galileo and Descartes of the modern mathematical understanding of the motion of free fall is seriously anachronistic. It supposes that the required concepts for a correct understanding—the modern concept of *instantaneous velocity*, of motion as a *function* relating varying instantaneous velocities to the *independent variable time*, of gravitational acceleration as truly *continuous*—were all available prior to the establishment of classical mechanics. As Damerow, Freudenthal, McLaughlin and Renn comment, such a "position either contains an implicit denial that conceptual development takes place at all, since the concepts remain the same as before, or else must claim that a conceptual development *preceded* the discovery of the central laws of a new science" (1992, 2). They argue that the classical concepts are "rather the outcomes of the establishment of the law [of free fall] than its prerequisites" (3).

This, however, is still to look at the history in terms of how it leads to the modern understanding of free fall. That is part of what interests us, of course. But once we peeled back the veil of assumed modern concepts, we found revealed a certain way of understanding motion with its own conceptual and mathematical trappings. If velocity is an affection of motion, and motion is a change of place occurring through some stretch of time, then there can be no such thing as an instantaneous velocity. Also, if velocity is the swiftness with which a particular motion is accomplished, then the velocities of two motions through equal spaces can be compared, and so can the velocities of two motions through equal times. For such a conception the theory of proportions seemed to be wholly adequate, and as we have seen, this is why neither Galileo (initially) and Descartes thought they could translate back and forth between the velocity of one motion being three times as fast as another through equal spaces to the first motion completing three times as much space as the second in equal times. Of course, this led to intractable difficulties when applied to nonuniform motion. But where these mathematical difficulties led Galileo to a realistic treatment of degrees of velocity, in which infinitely many of them "add" time-wise into an overall velocity, to his opponents these same difficulties, coupled with the difficulties of the composition of the continuum facing the Galilean approach, suggested that the only way successfully to treat free fall was in terms of discrete increments of uniform velocity. This, moreover, was in keeping with the mechanical philosophy, where all forces act by discrete tugs or pushes of microparticles, whereas the continuist account of acceleration lacked a plausible causal account.

Again from a retrospective point of view, one could conceive the episodes discussed above in terms of "obstacles" to progress, like the *obstacles épistémologiques* suggested by Gaston Bachelard. Thus so long as velocity was conceived as an affection of a motion achieved over time, scholars were prevented from forming the concept of instantaneous velocity; likewise, conceiving it as the swiftness with which an overall motion was accomplished, while enabling a treatment through the theory of proportions, was an impediment to a functional interpretation of motion, and to a successful treatment of non-uniform motion. But such a description still seems to presuppose a view of the history of mathematical physics as a linear progression to modern concepts taken as unassailable truths, the finishing line of the hurdler's race. One can say that the theory of proportions that went along with the Aristotelian conception of velocity was an obstacle to the formation of the modern concept only after that modern concept has been formed, but there is no guarantee that this is the only correct way to conceive motion, that there is a unique linear progression from the early seventeenth century to a perfectly correct modern concept. (Indeed, if we look at "motion with respect to cause" in a modern context, we again find the actions of continuous forces explained in terms of the collisions of micro-particles; and in this quantum context, a realistic picture is much more elusive than it was classically.)

I believe the situation is better captured by a notion I have introduced elsewhere (Arthur 2012), but which I have not yet fully developed. This is the notion of "epistemic vectors". These are aspects of theorizing, usually embodying a mathematical framework, that both impel and constrain thinking in a certain direction, without that implying that they have a known outcome or that they are necessarily an impediment. In that respect they differ from Bachelardian *obstacles*. They have something in common with Kuhn's idea of an exemplar, in that they may implicitly involve certain practices or ways of approaching problems. But they differ from exemplars in that they are implicit drivers of thought rather than exemplary components of an established paradigm that usually have a strong, explicit pedagogical function. Thus the law of free fall does become an exemplar after the further contributions of Huygens, Newton and Leibniz; but in the period we have been investigating, it has not yet become so, because the ideas constituting that exemplar are still in the process of being developed. The conception of velocity as the swiftness with which an overall motion is accomplished, together with the theory of proportions that accompanies it, is an epistemic vector; so is the conception of overall velocity as being representable by an area, with lines in that area representing degrees of velocity; so is the idea that changes in motion must be effected by the impacts of bodies. These vectors do not necessarily push theorizing in the same direction: the idea is that science progresses unevenly by the joint action of such vectors. Positing them helps to explain commonalities in the thinking of historical actors—for example, the fact that the young Galileo and Descartes both make what appears from hindsight to be the same mistake—as well as why what seem to us as right views were opposed—for example the opposition to Galileo's account of fall by many of his contemporaries, whom Drake regarded as merely benighted—why certain options did not occur to them, or why they persisted in proceeding down what appear to us as blind alleys. But I will leave a thorough explication of this notion of epistemic vectors for another occasion.

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