Leibniz's Actual Infinite in Relation to his Analysis of Matter*

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Abstract

In this paper I examine the relationship between Leibniz's thinking on the infinite and his analysis of matter. After contrasting his views on these subjects with those of Georg Cantor, I outline Leibniz's doctrine of the fictionality of infinite wholes and numbers by reference to his 1674 quadrature of the hyperbola, and defend its consistency against criticisms. In the third section I show how this same conception of the infinite informs Leibniz's thesis of the actually infinite division of matter. I defend his views on aggregation from Russell's criticism that they would make plurality a merely mental phenomenon, and expound Leibniz's argument that body is aggregated from unities that are not themselves parts of matter, although they are presupposed by them. I then argue that these unities of substance make actual the parts of matter, according to Leibniz, by being the foundation of the motions that individuate the actual parts of matter from one instant to another.

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Leibniz's Actual Infinite in Relation to his Analysis of Matter

I. *Introduction: the actually infinite division of matter*

It is well known that Leibniz held that matter is infinitely divided, and that there are infinitely many monads. But the connection between these two theses has not been well understood, and this has led to perplexity about Leibniz's views on the actual infinite and on the composition of matter, and also prompted accusations of inconsistency. Georg Cantor, for example, seized on Leibniz's endorsement of the actual infinite as an important precedent for this theory of the transfinite. Acknowledging that Leibniz "often pronounces himself against infinite number", Cantor declared that he was nevertheless "in the happy position of being able to cite pronouncements by the same thinker in which, to some extent in contradiction with himself, he expresses himself unequivocally *for* the actual infinite (as distinct from the Absolute)."[1] He quotes a typical passage to this effect from Leibniz's letter to Simon Foucher in 1692:

> I am so much in favour of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that she manifests it everywhere, the better to indicate the perfections of her Author. Thus I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and consequently the least particle ought to be considered as a world full of an infinity of different creatures.[2]

It should be remembered that Cantor was still battling with an Aristotelian orthodoxy in the philosophy of mathematics, a fact that explains the importance for him of Leibniz's claim that there are *actually infinitely many* created substances (monads), in defiance of the Aristotelian stricture that the infinite can exist only *potentially*.

Cantor, of course, believed that if there are actually infinitely many discrete creatures in any given part of matter, then there must be a corresponding infinite *number* of them. Indeed, his theory of transfinite numbers was designed to enable one to express precisely such statements. Now one might think that his success in establishing that mathematical theory would have marked the end of his interest in Leibniz's philosophy of matter and monads. That, however, is far from the case. In a side of his thought that perhaps deserves to be better known, Cantor did in

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1 Cantor, "Grundlagen einer allgemeinen Mannigfaltigkeitslehre" (1883), in Cantor 1932, 179. All translations in the present paper are my own. I cite current English translations where they exist for ease of reference.

2 Letter to Foucher, *Journal de Sçavans*, March 16, 1693, GP I 416 [= A II 2, N. 226. Leibniz an Simon Foucher (Wolfenbüttel, Ende Juni 1693), 713.].

fact use his theory of transfinite numbers to express such statements about the actually infinitely many constituents of matter. Declaring himself a follower of Leibniz's "organic philosophy", Cantor held "that in order to obtain a satisfactory *explanation of nature*, one must posit the ultimate or properly *simple* elements of matter to be *actually infinite in number*."[3] He continued:

> In agreement with Leibniz, I call these simple elements of nature *monads* or *unities*. [But since] there are *two specific, different types of matter interacting with one another*, namely *corporeal matter* and *aetherial matter*, one must also posit two different classes of *monads* as foundations, *corporeal monads* and *aetherial monads*. From this standpoint the question is raised (a question that occurred neither to Leibniz nor to later thinkers): what *power* is appropriate to these types of matter with respect to their elements, insofar as they are considered *sets* of *corporeal* and *aetherial monads*. In this connection, I frame the hypothesis that the *power [Mächtigkeit]* of the corporeal monads is (what I call in my researches) the *first* power, whilst the power of aetherial matter is the *second*. (Cantor 1932, 275-276)

That is, Cantor posits corporeal monads which, as discrete unities, are equinumerous with the natural numbers, and therefore have a power or cardinality $\aleph_0$, the first of the transfinite cardinal numbers. He further proposes that the aether, which he assumes to be continuous, is composed of aetherial monads equinumerous with the points on a line, i.e. that the number of aetherial monads is equal to $\aleph_1$, the second cardinal number, which he believed (but was never able to prove) to be the power of the continuum.[4]

There were certainly precedents for representing Leibniz's monads in this way as *elements* of matter. Euler, for instance, interpreted Leibniz's monads as "ultimate particles which enter into the composition of bodies" (Euler 1843, 39).[5] But modern commentators, equipped with a much more comprehensive selection of Leibniz's writings, would not consider Cantor's

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3 This and the quotations following are culled from Cantor's "*Über verschiedene Theoreme aus der Theorie der Punktmengen in einem n-fach ausgedehnten stetigen Raume G_n*" (1885), (Cantor 1932, 275-276).

4 Cantor 1932, 276. Cf. Dauben 1979, 292. Since Cantor's time it has been shown that his Continuum Hypothesis is consistent with (by Gödel, in 1940) but independent of (by Cohen, in 1963) Zermelo-Fraenkel set theory, the standard foundation of modern mathematics, provided ZF set theory is consistent.

5 Euler also called Leibniz's monads *parts* of bodies that result from a "limited division" (1843, 48-49). Perhaps also contributing to these eighteenth century misunderstandings of Leibniz were the views of Maupertuis, where the fundamental particles of matter are physical points endowed with appetition and perception, and of Boscovic, whose particles are interacting point-sources of attractive and repulsive force.

reinterpretation of Leibnizian monads a faithful elaboration of his views. For Leibniz conceived his monads or simple substances neither as *interacting* with one another, nor as *elements* out of which matter is composed. Both these ideas are closer to the views of Christian Wolff, who advocated a theory of simple substances that were avowedly not monads in Leibniz's sense. Instead, Wolff's substances were supposed to give rise to material atoms through their interactions, which atoms were the elements out of which extended bodies were then aggregated.[6]

Putting those objections to one side, however, modern scholars have by and large taken Cantor's side on the actual infinite, agreeing with his criticisms of Leibniz's rejection of infinite number. They point out that Leibniz would have had a precedent for embracing infinite number in Galileo Galilei, who in his *Two New Sciences* had declared matter to be composed of an actually infinite number of atoms, separated by infinitely small voids. Had Leibniz followed Galileo's lead, it is often asserted, he might have well anticipated Cantor's transfinite.[7]

I am not convinced, however, that such claims are justified. Leibniz insisted that the axiom that the whole is greater than its (proper) part must hold in the infinite as well as in the finite, whereas Cantor followed Dedekind in maintaining that an infinity of elements could be equal in number to an (infinite) proper subset of those elements, and consequently denied the applicability of the part-whole axiom to infinite sets. Galileo, by contrast, denied that the notions of being 'greater', 'equal to' or 'less than' applied in the infinite at all, thus contradicting both Leibniz and Cantor. But this very denial made his approach unsuitable as a foundation for the mathematics of the infinite.[8]

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6 Wolff's simple substances should not, however, be conflated with the (extended) primitive corpuscles which he supposed were constituted by their interactions. For whereas these primitive corpuscles are finitely extended and only physically indivisible, Wolff's simple substances are unextended because partless, and aggregate into extended bodies. See (Wolff 1737, §182, 186, 187). A succinct and accurate summary of these aspects of Wolff's philosophy is given by Matt Hettche in his (2006).

7 Rescher states that the Cantorian theory of the transfinite, point-set topology and measure theory "have shown that Leibniz's method of attack was poor. Indeed, Galileo had already handled the problem more satisfactorily ..." (Rescher 1967, 111); Gregory Brown asserts that, "had he not jumped the gun in rejecting the possibility of infinite number and infinite wholes, Leibniz, having already surmounted the prejudice against actual infinities, would have been well placed to anticipate the discoveries of Cantor and Frege by at least two hundred years." (Brown 2000, 24; see also his 1998, 122-123).

8 For Galileo the infinitely small parts of the continuum are *non quante*, whereas Leibniz insisted from the beginning that they were quantifiable. Leibniz continued to maintain that the infinite and the infinitely small are quantifiable even after defining them as useful fictions, which is why they are treatable in his differential calculus. I have argued this at greater length in Arthur 2001.

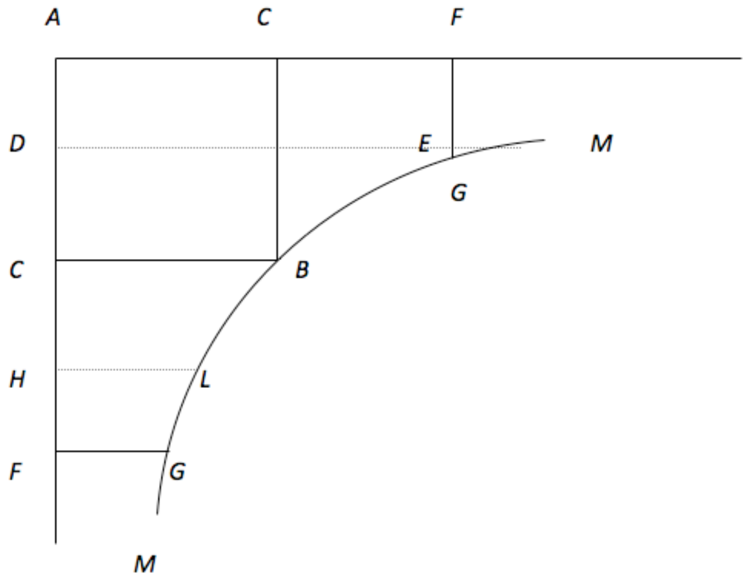
II. *The fictionality of infinite wholes and collections*

Leibniz, in fact, was well aware of Galileo's argument in the *Two New Sciences* that "in the infinite there is neither greater nor smaller", having himself made notes on it in the Fall of 1672 (A VI 3, 168; LoC 6-9). He gives Galileo's demonstration as follows:

> Among numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares as numbers, that is to say, there are as many square numbers as there are numbers in the universe. Which is impossible. Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one and not a whole. (LoC 9)

Leibniz had made these notes soon after his arrival in Paris, by which time he was already at work on infinite series under Huygens' guidance. These studies in proto-calculus did nothing to dissuade him from this opinion about the infinite not being a true whole. In fact, the expressions he found for the area under a hyperbola in terms of infinite series gave a pictorial representation of an infinite whole being equal to its part, and at the same time seemed to him to confirm the fictional nature of such a whole.

The following is a condensed version of Leibniz's argument from a paper written in October 1674 (A VII 3, 468). He gives a symmetrical diagram of a hyperbola with centre A, vertex B, and radius $AC = BC = a$, which, without loss of generality, may be set equal to 1. M represents the fictional point where the curve "meets" each line AF... at infinity. Here the x-axis is the line ACF... across the top of the figure, the y-axis runs with increasing y down the page from A through DCH to F, and the dotted lines DE and HL represent the variable abscissa x. Thus $AD = AC - CD = 1 - y$, and $AH = AC + CH = AC + CD = 1 + y$.



Leibniz now sets about discovering the area under the curve between the horizontal line *CB* and the *x*-axis running across the top of the figure. We have $DE = 1/AD = 1/(1 - y)$, which he expands as an infinite series:

$$DE = 1/(1 - y) = 1 + y + y^2 + y^3 + y^4 + y^5 + \ldots$$

Now the area in question is obtained by "applying" the variable line *DE* to the line $AC = 1$, giving

$$\text{Area}(ACBEM) = 1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + {}^1/_6 \ldots$$

In modern terms, Leibniz has *integrated* a power series expansion of $1/(1 - y)$ with respect to y, $\int(1 + y + y^2 + y^3 \ \ldots)\, dy$, between 0 and 1. By a similar argument, he obtains

$$\text{Area}(CFGLB) = 1 - {}^1/_2 + {}^1/_3 - {}^1/_4 + {}^1/_5 - {}^1/_6 \text{ etc.} = {}^1/_2 + {}^1/_{12} + {}^1/_{30} + {}^1/_{56} + {}^1/_{90} \ldots$$

Now Leibniz subtracts this finite area *CFGLB* from the infinite area *ACBEM*, to get

$$\text{Area}(ACBEM - CFGLB) = 1 - (1) + {}^1/_2 - (-{}^1/_2) + {}^1/_3 - (+{}^1/_3) + {}^1/_4 - (-{}^1/_4) + {}^1/_5 - (+{}^1/_5) \ldots$$

$$= {}^2/_2 + {}^2/_4 + {}^2/_6 + {}^2/_8 + {}^2/_{10} + {}^2/_{12} \ldots$$

$$= 1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + {}^1/_6 \ldots$$

$$= \text{Area}(ACBEM)$$

That is, subtracting the area *CFGLB*—an area that is perfectly definite and observable—from the area under the curve leaves that area the same! Leibniz comments:

> This is remarkable enough, and shows that the sum of the series 1, $^1/_2$, $^1/_3$, etc. is infinite, and therefore that the area of the space *ACGBM*, even when the finite space *CBGF* is taken away from it, remains the same, i.e. this takes away nothing observable (*notabile*). By this argument it can be concluded that the infinite is not a whole, but a fiction, since otherwise the part would be equal to the whole. (A VII 3, 468)

It is noteworthy here that Leibniz concludes that the sum of the series $1+ {}^1/_2 + {}^1/_3 + \ldots$ is infinite on the implicit grounds that, if it is a whole, it is equal to a proper part of itself. As is well known, Dedekind, having followed Bolzano in characterizing a set as infinite if it can be set in 1-1 correspondence with a proper subset of itself, defines two sets as equal iff there exists a 1-1 correspondence (bijection) between their elements—the definition of equality of sets that Cantor takes as foundational for his theory of size of infinite sets.[9] As Russell remarks in his *Introduction to Mathematical Philosophy* (1919, 81), this enables Cantor to avoid Galileo's Paradox. For if one takes the criterion of bijection as defining the equality ("similarity" in his terminology) of infinite sets, then "there is no contradiction, since an infinite collection can perfectly well have parts similar to itself". That is, assuming infinite collections and the Dedekindian definition of equality adopted by Cantor, it can no longer be maintained that it is "self-contradictory that 'the part should be equal to the whole'" (80-81). The contradiction is avoided by jettisoning the part-whole axiom (P) rather than the assumption that an infinity of terms can be collected into one whole (C).

As I have argued elsewhere, this is typical of a reductio argument, where the conclusion to draw from proving a set of premises inconsistent depends on which premise one is prepared to reject. Leibniz, insisting on P, infers ¬C. Russell and Cantor, insisting on C, infer ¬P (Arthur, 2001, 103). Gregory Brown has taken issue with this logic, arguing that, "given the consistency of Cantorian set theory, it would again appear that Leibniz's argument against infinite number and infinite wholes must be unsound." (Brown 2005, 481). He argues that if Leibniz's argument

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9 Although Bernard Bolzano recognized that it is a characteristic property of an infinite set that its members may be set in 1-1 correspondence with those of a proper subset of itself, he denied that the two sets would therefore have the same number of members. That was Dedekind's contribution. See Mancosu 2009, esp. pp. 624-627 for an illuminating discussion.

were sound, then Cantor's theory could not even be consistent, because it is erected on definitions of 'less than', 'greater than', and 'equal to'—"given in terms of one–one correspondence of sets" (484)—that are in contradiction with Leibniz's. But this is just to claim that, *given C*, P cannot be taken to be axiomatically true.

In mounting his case against Leibniz, Brown appeals to an argument given by Jose Benardete in his book on infinity (1964), who accused Leibniz of equivocating between different senses of the concept of equality—a criticism originating with Russell in his (1919). Benardete's criticism has recently been developed further by Mark van Atten, who argues that "Russell and others have observed that Leibniz's argument is not correct because it rests on an equivocation on the concept of equality." (van Atten 2011, 123). Van Atten makes his case by quoting several versions of Leibniz's argument, and then giving a reconstruction in which the third line is "3. The multitude of the squares is equal to a part of the whole of the numbers", and the fifth is "5. The multitude of the squares is equal to the whole of the numbers." "Clearly," he writes, "in line 3 'is equal to' means 'is identical to', while in line 5 it means 'can be put in a bijection with'" (123).

I do not agree that Leibniz is guilty of such an equivocation. In all versions he gives of this argument he accepts that if there is a bijection between the terms in two multiplicities (*multitudines*),[10] then they are equal. That is, Leibniz accepts that bijection implies equality; he does not have to accept it as a definition of equality. Van Atten uses against him his definition of sameness or coincidence of terms as those "which can be substituted for each other without affecting the truth" to show that Leibniz uses "equality" in two diverse senses: "the concepts of equality in lines 3 and 5 are diverse or different, as substitution of the one for the other does not preserve truth here" (124). But the only criterion of equality Leibniz appeals to is that of bijection: if the even numbers have a 1-1 correspondence with a subset of the whole numbers, the multiplicity of even numbers is equal to a part of the multiplicity of whole numbers. Here by "part", Leibniz understands "proper part" or proper subset, as van Atten acknowledges, so that one multiplicity B is a part of another multiplicity A if there are no terms in B that are not in A,

10 Leibniz's neutral term for a plurality is *multitudo*, which Van Atten and others translate literally as "multitude", whereas I follow Russell in translating it as *multiplicity*. Unfortunately, Leibniz is not always consistent in his use of terms, sometimes using *collectio* (collection) as a synonym. But he wants to deny infinite collections, while still maintaining that there are infinite multiplicities or aggregates: one can collect together all the terms of a multiplicity if it is finite, but not if it is infinite.

while there are terms in A that are not in B. It will then follow that, if the multiplicity of natural numbers N and that of square numbers Q can be treated as wholes, Q is a part of the whole N; but since there is a 1-1 correspondence between their terms, they are equal: the whole is equal to the part. In a letter to Justus Christoph Böhmer in June, 1694, Leibniz phrases the argument in exactly this way:

> If any A has a B corresponding to it, and any B has an A corresponding to it; it follows that there are as many A as there are B and vice versa.
> But any number has a square corresponding to it, and any square has a number corresponding to it.
> Therefore there are as many numbers as there are squares, and vice versa.
> Therefore the multiplicity of all numbers (if such there is), is equal to the multiplicity of all squares.
> But the multiplicity of all numbers is a whole, and the multiplicity of all squares is a part, because the multiplicity of all numbers contains numbers that are not squares as well.
> Therefore the whole is equal to the part. Which is absurd.
> Therefore there is no multiplicity or number of all numbers, nor of all squares, but rather such a thing is chimerical. (A II 2B, 814)

Thus, given the part-whole axiom P (and his interpretation of 'part'), Leibniz is correct to infer ¬C, that an infinite collection of terms does not constitute a true whole. And this denial of the existence of infinite collections is equivalent, as accepted on all sides, to the denial of infinite number.

Now it is true that Leibniz claimed that he could *demonstrate* P from other notions, such as "B is B" and "each thing is equal to itself", as Brown reminds us in his criticisms of Leibniz's argument. But the very consistency of Cantor's theory of the transfinite, he argues, shows that Leibniz must have been wrong to claim this (Brown 2005, 484).[11] If Leibniz had indeed demonstrated it, then Cantor's theory would be unsound. Exactly this kind of criticism has been subjected to a detailed rebuttal by Herbert Breger in his (2008), in response to Van Atten's

[11] Of course, one could also object to Brown that Cantor's own "naive" set theory is known to be *in*consistent; and that while it has now been replaced as the foundation of mathematics by Zermelo-Fraenkel set theory with the axiom of choice (ZFC), the consistency of ZFC (as Brown himself acknowledges on p. 486) has *not* been established yet. It is simply assumed that if ZFC were inconsistent, a contradiction in it would have been discovered by now.

criticisms. An argument of this kind, Breger argues, can show that Leibniz was wrong *only if one assumes that there is just one correct foundation for mathematics, and that that was given by Cantor*. Such a conception of mathematics, Breger argues, is belied by its actual historical development.[12] It is clear, he argues, that each decision to generalize concepts of mathematics—such as the decision to admit negative numbers, imaginary numbers, or the transfinite—involved a contingent decision, made on philosophical grounds. One can argue that the revised definition is superior on the basis of the mathematics it enables, but it is a historiographical mistake then to interpret *in terms of this revised definition* statements made about numbers prior to the acceptance of this definition, in order to show that these statements are false. On Leibniz's conception, the part-whole axiom is constitutive of number just because no comparisons of quantity can be made without presupposing it. If bijection implies equality, this entails that infinite number entails a contradiction, given P. One cannot find fault with this argument by giving transfinite numbers as a counterexample, since these are numbers only in the Cantorian revised sense of number based on the existence of infinite collections, for which P fails: they are not instances of the numbers Leibniz is discussing.[13]

Breger's point about the contingency of the historical development of mathematics, as well as his and my defence of the tenability of Leibniz's retaining the part-whole axiom, have recently received strong indirect support from an intriguing article by Paolo Mancosu on measuring the size of infinite collections (Mancosu 2009). Mancosu traces the idea that the size of an infinite set can be based on the part-whole axiom from its historical roots to recent developments in mathematics. He uses the existence of alternative theories of class sizes based on the part-whole axiom (due to Katz, Benci, Di Nasso and Forti) to "debunk" arguments such as Kurt Gödel's that the Cantorian conception of infinite number (based on the Dedekindian definition of the equality of sets) is inevitable.[14] These theories of class size, of course, depend on the existence of infinite

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12 "One must renounce the assumption … that there is *one* mathematics — an assumption that should in fact have gone out of date with the acceptance of non-Euclidean geometries." (Breger 2008, 314).

13 Cf. Breger's discussion of the two notions of 'being the same number', (i) A = B iff neither A nor B is a proper part of the other, and (ii) A = B iff there is a bivalent mapping between them: "The fact the one finds objects outside the theory examined here for which both notions are not equivalent is of no importance within the theory." (2008, 314)

14 "I have hoped to show that the possibility of comparing Cantor's theory against the alternative theories of class sizes (CS) and numerosities allows us to analyze more finely, and in some cases debunk, the arguments that claim either the inevitability of the Cantorian choice (Gödel) or that account for the (alleged) explanatory nature of the

collections, which Leibniz denies, so they are not compatible with Leibniz's own approach. But their very existence shows that there is nothing inevitable about having to adopt the Dedekindian definition of the equality of sets or about having to reject the part-whole axiom.

Furthermore, despite his rejection of infinite number and infinite collections, Leibniz maintained that one can make true statements about an infinity of terms, such as in asserting that there are infinitely many terms in an infinite series. This does not mean that there is an infinite *number* of terms. It means that there are more terms than can be assigned any finite number. As Leibniz explained to Johann Bernoulli in a letter of February 21, 1699,

> We can conceive an infinite series consisting merely of finite terms, or terms ordered in a decreasing geometric progression. I concede the infinite multiplicity of terms, but this multiplicity forms neither a number nor one whole. It means only that there are more terms than can be designated by a number; just as there is for instance a multiplicity or complex of all numbers; but this multiplicity is neither a number nor one whole. (GM III 575)

This is Leibniz's actual but *syncategorematic* infinite. This term alludes to the distinction first formulated by Peter of Spain, and later elaborated by Jean Buridan, Gregory of Rimini and William of Ockham, who claimed that to assert that the continuum has infinitely many parts in a *syncategorematic* sense is to assert that "there are not so many parts finite in number that there are not more (*partes non tot finitas numero quin plures,* or *non sunt tot quin sint plura*)". The statement is called *syncategorematic* because the term 'infinite' occurs in it, but that term does not actually have a referent corresponding to it.[15] Rather, it gains its meaning from the way the statement as a whole functions. This is contrasted with the *categorematic* sense of infinity, according to which to say that there are infinitely many parts is to say that there is a number of parts greater than any finite number, i.e. that there is an infinite number of parts. As Leibniz elaborates in his *New Essays*,

> It is perfectly correct to say that there is an infinity of things, i.e. that there are always more of them than can be specified. But it is easy to demonstrate that there is no infinite

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Cantorian generalization by appealing to the (alleged) nonrational nature of preserving the part–whole principle." (Mancosu 2009, 642)

15 See O. B. Bassler's erudite footnote on the syncategorematic and categorematic in his 1998, 855, n. 15, and the references cited therein. See also Sam Levey's 2008 for a careful elaboration of Leibniz's fictionalism.

> number, nor any infinite line or other infinite quantity, if these are taken to be genuine wholes. The Scholastics were taking that view, or should have been doing so, when they allowed a 'syncategorematic' infinite, as they called it, but not a 'categorematic' one. (Leibniz, *New Essays*, 1981, §157; GP V 144; Russell 1900, 244)

Leibniz's actual but syncategorematic infinite is thus distinct from Aristotle's potential infinite, in that it embraces an infinity of actually existents, but it also differs profoundly from Cantor's theory of the actual infinite as transfinite in that it denies the existence of infinite collections or sets that are the basis of transfinite set theory. It is not the existence of an infinite plurality of terms that is denied, but the existence of an infinite collection, and thus an infinite number, of them. For instance, to assert that *there are infinitely many primes*, is to assert that, for any finite number x that you choose to number the primes, there is a number of primes y greater than this: $\forall x \exists y(\mathrm{F}x \rightarrow y > x)$, where $\mathrm{F}x$:= 'x is finite', and x and y are any numbers. By contrast, to assert their infinity *categorematically* would be to assert that there exists some one number of primes y which is greater than any finite number x, i.e. that $\exists y \forall x(\mathrm{F}x \rightarrow y > x)$ —i. e. that there exists an infinite number. Interestingly, Euclid's proof that there are infinitely many primes begins by assuming that there finitely many, so that there is a greatest prime, and then deriving a contradiction. But the negation of the assumption that there is a greatest finite prime, $\exists x \forall y[\mathrm{F}x \,\&\, (y \neq x \rightarrow y \leq x)]$, is provably equivalent to $\forall x \exists y[\mathrm{F}x \rightarrow (y \neq x \,\&\, y > x)]$. This says "for any finite number of primes, there is a number of primes different from and greater than this". This is the syncategorematic actual infinite, not the categorematic, which would be $\exists y \forall x[\mathrm{F}x \rightarrow (y \neq x \,\&\, y > x)]$: "there exists a number of primes greater than any finite number".

As I have argued elsewhere (Arthur 2008), this provides the foundation for a surprisingly cogent theory of the infinite and infinitesimal. Leibniz had already recognized in 1674 (as can be seen by the above quotation) that the infinite can be treated as a *fiction*. This signifies that it can be treated *as if* it is an entity, in a certain respect, provided that the statements in which it occurs can be interpreted without supposing there is such a thing. For instance, in an infinite series, the infinite multiplicity of terms can be treated as if they are a collection of terms added together, provided a workable account of this infinite 'sum' can be given which does not presume this. In 1676, Leibniz finds a definition of the sum of a converging infinite series in keeping with his syncategorematic account of the infinite:

> Whenever it is said that a certain infinite series of numbers has a sum, I am of the opinion that all that is being said is that any finite series with the same rule has a sum, and that the error always diminishes as the series increases, so that it becomes as small as we would like. For numbers do not *in themselves* go absolutely to infinity, since then there would be a greatest number. (A VI 3, 503; LoC 98-99)

This is a good example of how Leibniz's philosophy of the actual infinite is supposed to work: you can still do mathematics with infinite quantities. Under certain conditions, they can be treated as fictional wholes, in the same way that the sum of this infinite series is a fictional sum, and the justification is in terms that, after Cauchy, we would now express in terms of ε and δ. Leibniz's definition here of the sum of a converging infinite series is equivalent to the modern one in terms of a limit of partial sums, and does not involve first taking an actual infinity of terms and then forming a sum of them.[16]

III. *The Leibnizian analysis of matter*

Turning now to the analysis of matter, we find the kind of picture that Cantor was proposing where continuous aetherial matter would be composed out of an (uncountable) infinity of substances ruled out in principle. The continuum, Leibniz claims, is something ideal, whereas what is real is an aggregate of unities. Here are some typical aphorisms to this effect:

> In actuals, simples are prior to aggregates, in ideals the whole is prior to the part. The neglect of this consideration has brought forth the labyrinth of the continuum. (To Des Bosses, 31st July 1709; GP II 379; Russell 245)
>
> Actuals are composed as is a number out of unities, ideals as a number out of fractions: the parts are actual in the real whole, not in the ideal whole. In fact we are confusing ideals with real substances when we seek actual parts in the order of possibles, and

16 As this paper goes to print, my attention has been drawn to a paper by David Rabouin (2011), who gives a reading entirely compatible with the present one by comparing Leibniz's philosophy of the infinite with that of Nicholas of Cusa. See also Ishiguro (1990), who was one of the first to argue that Leibniz can allow for the success of treating the infinite and infinitely small *as if* they are entities (under certain conditions), and that it is this that allows him to claim that mathematical practice is not affected by whether one takes them to be real or not. Philip Beeley (2009) gives a subtly different reading, interpreting such claims as instances of Leibniz's pragmatism. His paper is highly recommended for the intriguing connections he traces between infinity, conceptual analysis, the divine mind and the universal characteristic in the development of Leibniz's thought.

> indeterminate parts in the aggregate of actuals, and we entangle ourselves in the labyrinth of the continuum and inexplicable contradictions. (To De Volder, 19/1/1706: GP II 282)
>
> It is the confusion of the ideal and the actual that has embroiled everything and produced the labyrinth *of the composition of the continuum*. Those who compose a line from points have quite improperly sought first elements in ideal things or relations; and those who have found that relations such as number and space (which comprise the order or relation of possible coexistent things) cannot be formed from an assemblage of points, have for the most part been mistaken in denying that substantial realities have first elements, as if there were no primitive unities in them, or as if there were no simple substances. (Remark on Foucher's Objections (1695); GP IV 491)

On the basis of such claims, Nicholas Rescher has concluded that the solution Leibniz is offering to the continuum problem is that in the mathematical continuum the whole is prior to the parts, but in the metaphysical one, the parts (monads) are prior to the whole. For it is not case that "both the indivisible constituent and the continuum to which it belongs [can] both at once be real":

> In mathematics the continuum, the line, is real and the point is merely the ideal limit of an infinite subdivision. In metaphysics only the ultimate constituents, the monads, are actual, and any continuum to which they give rise is but phenomenal. This is the Leibnizian solution of the paradoxes of the continuum. (Rescher 1967, 111)

Thus Rescher attributes to Leibniz a two-tiered ontology: the metaphysical, in which the actual monads are constituents of a phenomenal continuum, and the mathematical, in which the line segments are real and points are their merely ideal limits.

There are many problems with this analysis, however. Regarding the mathematical, Leibniz is clear that all mathematical objects are *ideal entities*, and (after 1676) that a point is always an *endpoint* of a line segment, never the ideal limit of an infinite subdivision. The infinitesimals of his mature theory, moreover, are *not* indivisible, and are in any case fictions rather than actual parts. And conversely, on the metaphysical side, any phenomenal whole resulting from an aggregate of monads ("secondary matter" or "body") must be well founded or *real,* and cannot therefore have the indeterminate parts characteristic of the continuous: "But in real things, namely bodies, the parts are not indefinite, as they are in space, a mental thing" (To De Volder, 6/30/1704: GP II 268); "in actuals there is nothing indefinite—indeed in them every division that can be made is made ... the parts are actual in the real whole ..." (To De Volder, 1st January 1706: GP II 282). Consequently such a real phenomenal whole is *discrete* rather than a continuum: "Matter is not continuous, but discrete ... [It is the same with changes, which are not really continuous.]" (To De Volder, 11th October 1705; GP II 278, R 245).[17]

Many of these criticisms were made by J. E. McGuire in his (1976), who consequently ascribed to Leibniz a three-tiered ontology, consisting in the actual, the phenomenal and the ideal.[18] In distinguishing these levels he makes use of the distinction in Leibniz between division and resolution, a distinction of crucial importance in understanding Leibniz's views, as we shall see in due course. Thus in Leibniz's mature metaphysics, space and time are characterized as ideal, or "entities of reason" (*entia rationis*) (McGuire 1976, 307; Hartz and Cover, 1988, 504, 513). As continuous entities, they are arbitrarily divisible, though not composed of parts (McGuire 309; Hartz and Cover 505). They are *resolvable* into points or instants, but neither divisible into nor composed out of these, since points and instants are mere modalities (McGuire 1976, 309-310). Well-founded phenomena, on the other hand are extended aggregates, and as such presuppose a plurality of entities from which their extension results. Thus they are *resolvable* into units of substance. They are also *divisible* into actual parts, and *composed* of these parts. McGuire concludes, however, that the only *actuals* are the substantial unities into which phenomena are resolved. Being simple, these actual substances themselves "can be neither composable, nor resolvable, nor divisible" (310).

The difficulty with this last claim is as follows. If monads or simple substances are the only actuals, then they must be the actual parts from which phenomena are composed, as McGuire duly concludes: "the 'actual parts' of extended things are non-extended substances" (306). But

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17 On an Aristotelian understanding of the continuous, a continuum is unbroken, and has no actual boundaries within. A line that is actually divided into contiguous line segments is therefore no longer regarded as continuous but as possessing discrete parts, notwithstanding the fact that there are no gaps between these contiguous segments. Thus when Leibniz describes matter as "discrete" he means actually divided into contiguous parts, but as still forming a plenum. For an engaging and informative history of this conception of the continuum as cohesive and unbroken, from Aristotle to present-day smooth infinitesimal analysis, see Bell 2006.

18 Here he has been followed by Glenn Hartz and Jan Cover (1988), who contend that Leibniz changed his position from a 2-realm view to the 3-realm view of his mature metaphysics, after a period of transition between the years 1695 and 1709.

this flies in the face of what Leibniz says, as McGuire is perhaps tacitly acknowledging by his use of scare quotes.[19] Monads are supposed to be simples, entities into which bodies and motions are *resolved*, not parts out of which they are *composed*. As Leibniz writes to Burcher de Volder in 1704 in a typical passage,

> But, accurately speaking, matter is not composed of constitutive unities, but results from them, since matter or extended mass is nothing but a phenomenon founded in things, like a rainbow or mock-sun, and all reality belongs only to unities ... Substantial unities, in fact, are not parts but foundations of phenomena. (To De Volder, 30th June, 1704: GP II 268)

The actual parts of phenomenal bodies, in fact, are not substances, but actually existing parts, as opposed to the indefinite parts into which a continuous body is divisible. Indeed, they are always mentioned by Leibniz in the context of an *actual division.* In the letter to De Volder of 19/1/1706, Leibniz writes "in actuals there is nothing indefinite—indeed, in them any division that can be made, is made" (G II 282). And in his Remarks of 1695, the points marking the possible divisions of an abstract line are contrasted with the "the divisions actually made, which designate these points in an entirely different manner" (G IV 49). Now since only phenomenal bodies can be divided (simple substances are indivisible), this means that Leibniz's "actual parts" must be parts of actually divided phenomenal bodies; and these parts will again be bodies. This is confirmed more explicitly in the following passages:

> But in real things, that is, bodies, the parts are not indefinite (as they are in space, a mental thing), but are actually assigned in a certain way, as nature actually institutes the divisions and subdivisions according to the varieties of motion, and ... these divisions proceed to infinity... (to De Volder, June 30th, 1704: G II 268).

> We should think of space as full of matter which is inherently fluid, capable of every sort of division, and indeed actually divided and subdivided to infinity; but with this difference, that how it is divisible and divided varies from place to place, because of

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[19] Although Hartz and Cover criticize McGuire for his "misuse ... of 'actual' to distinguish monads from bodies" (1988, 519), they nonetheless assert that "extension conceived as an abstract continuum has no actual parts, but extended bodies do have such parts: they are the genuine composites whose actual parts are Leibniz's 'atoms of substance' (cf. L 539, G II 282)." (1988, p. 497).

> variations in the extent to which the movements in it run the same way. (*New Essays*, Preface; Leibniz 1981, 59)

As these passages indicate, the actual divisions of matter are determined by the "varieties of motion". This is premised on the idea that any given body is individuated by its parts all having a motion in common, so that parts with differing motions will be actually divided from one another, as Descartes had in fact argued in his *Principles of Philosophy* (II, §34-35). But in a plenum, according to Leibniz, every body is acted upon by those around it, causing differentiated motions in its interior, and thus dividing it. Since this is the case for every body, the division will proceed to infinity. Therefore matter is actually infinitely divided. Moreover, since what is divided is an aggregate of the parts into which it is divided, it will be an *infinite aggregate.* This argument is stated by Leibniz on numerous occasions throughout his *oeuvre*, including in the *Monadology* (1714): "every portion of matter is not only divisible to infinity, as the ancients realized, but is actually subdivided without end, every part into smaller parts, each one of which has its own motion." (WFPT 277). A particularly explicit example occurs in an unpublished fragment probably dating from 1678-9:

> Created things are actually infinite. For any body whatever is actually divided into several parts, since any body whatever is acted upon by other bodies. And any part whatever of a body is a body by the very definition of body. So bodies are actually infinite, i.e. more bodies can be found than there are unities in any given number. (c. 1678-9; A VI 4, 1393; LoC 235).

Significantly, the notion of the actual infinite Leibniz appeals to here is precisely the *syncategorematic* notion explained above: bodies are actually infinite in the sense that for any finite number, there are actually (not merely potentially) more bodies than this. Regarding the actual infinite, then, there is a perfect consilience between Leibniz's mathematics and his natural philosophy.

One consequence of this is that a body cannot be a true whole. For since every body is an infinite aggregate of its parts, it is an infinite whole; and, as we have seen, Leibniz held infinite wholes to be fictions. In an earlier work (Arthur 1989) I suggested that this explains in part why Leibniz held that bodies are phenomenal: since he regarded any substance as a true unity, bodies, being only fictional unities, would not qualify as substances. If a phenomenon is something that appears to the senses but is not a substance, then bodies, insofar as they really appear to the senses, must qualify as real phenomena. Thus the fact that bodies are phenomena is explained in part by Leibniz's doctrine of the actual infinite.

To this, two objections can be made. First, as Gregory Brown objected, a unity is not the same as a whole (Brown 2000, 41). Since Leibniz held that no substance can be composed of parts, no substance can be a whole, whether fictional or true.[20] Secondly, as Russell had already perceptively observed in 1900, although it is true that Leibniz identified bodies as infinite aggregates, and these as "corresponding to the phenomena", he also claimed that *all* aggregates are phenomenal, even finite ones: "A collection of substances does not really constitute a true substance. It is something resultant, which is given its final touch of unity by the soul's thought and perception." (Leibniz, *New Essays*, 1981, 226). Thus, as Russell remarks, "even a finite aggregate of monads is not a whole per se. The unity is mental or semi-mental" (Russell 1900, 116). This conclusion is a consequence of Leibniz's nominalism about aggregates, again accurately epitomised by Russell: "Whatever is real about an aggregate is *only* the reality of its constituents taken one at a time; the unity of a collection is what Leibniz calls semi-mental (GP II 304), and therefore the collection is phenomenal although its constituents are all real." (115)

When Russell was writing his book on Leibniz in 1900, he had still not encountered Cantor's set theory and was sympathetic to Leibniz's doctrine "that infinite aggregates have no number", describing it as "one of the best ways of escaping from the antinomy of infinite number" (117). But whatever reservations he may have harboured then about number in connection with infinite aggregates, he certainly had none about finite aggregates, and believed the doctrine of the phenomenality of aggregates to be a serious deficiency in Leibniz's philosophy. Russell took this doctrine to be a consequence of Leibniz's deriving his metaphysics from his logic, as "the assertion of a plurality of substances is not of this [subject-predicate] form—it does not assign predicates to a substance" (116).) But, he argued, if it is "the mind, and the mind only, [that] synthesizes the diversity of monads", then "a collection, as such, acquires only a precarious and derived reality from simultaneous perception" (116). So he confronted Leibniz with a dilemma:

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[20] Here I owe a profound debt to my former student Adam Harmer, who first persuaded me of the significance of Leibniz's "mereological nihilism" for his notion of corporeal substance: this cannot be, as I had formerly supposed, a substance with a body that is a true whole at any given time.

> For the present it is enough to place a dilemma before Leibniz. If the plurality lies *only* in the percipient, there cannot be many percipients, and the whole doctrine of monads collapses. If the plurality lies *not* only in the percipient, then there is a proposition not reducible to the subject-predicate form, the basis for the use of substance has fallen through, and the assertion of infinite aggregates, with all its contradictions, becomes quite inevitable for Leibniz. The boasted solution of the difficulties of the continuum is thus resolved into smoke, and we are left with all the problems of matter unanswered. (Russell 1900, 117)

Even if we set aside Russell's mistaken belief that Leibniz derived his metaphysics of substance from a commitment to subject-predicate logic, though, there still remains a dilemma, given his reading of Leibniz's stance on plurality. For if plurality lies only in the percipient, then Leibniz is not entitled to assert that there is objectively more than one substance, and his system collapses into a monism; the infinite plurality of parts into which matter is divided must likewise exist only in the mind. Whereas if plurality is mind-independent, this seems to deprive Leibniz of any ground for denying the principle that to every aggregate there corresponds a number. In that case, Russell suggests, Leibniz would be forced to concede that there is infinite number, and he would fall into the very antinomies he was trying to avoid.

In fact, however, Russell's dilemma is based on a mistaken reading of Leibniz's doctrine of aggregates. It is not the plurality or aggregate that lies only in the percipient, but the perception of it *as a unity*. It is not the plurality itself that is contributed by perception, but the aggregate conceived as an entity *distinct from* its constituents. The *reality* of the aggregate does indeed consist in that of the constituents of the aggregate, just as Russell had described. What that means however is that if one has a flock of twenty sheep, say, each of these sheep exists independently of anyone perceiving it. If they are conceived or perceived together as making up a flock, then, according to Leibniz's doctrine, it is the flock *as distinct from its constituents* whose existence consists in those constituents being conceived or perceived together. This is consistent with Leibniz's position that numbers are ideal entities: the multiplicity of sheep exists independently of anyone numbering, but the numbering of the sheep as twenty requires someone to conceive them as making up a score or viguple, as a twenty. Thus the plurality itself is not mind-dependent, but only the judgement of it as forming a unity is, as is the applying to it of a number. In the same way, the divisions in matter are actual: they are not the result of any mental judgement, but of the internal motions of matter which are responsible for the divisions.

Leibniz is perfectly explicit on this point, as for instance in this passage from a letter to De Volder:

> I think that that which is extended has no unity except in the abstract, namely when we divert the mind from the internal motion of the parts by which each and every part of matter is, in turn, actually subdivided into different parts, something that plenitude does not prevent. Nor do the parts of matter differ only modally if they are divided by souls and entelechies, which always persist. (to De Volder, 3 April 1699; G II 282)

Why, then, does Leibniz insist that matter is phenomenal? Again, his argument is laid out quite explicitly, both to Arnauld and to De Volder. It depends, as Russell recognized, on a "very bold use" (Russell 1900, 115) of his nominalist principle that *the reality of an aggregate derives only from the reality of its constituents*. As he explained to Arnauld, it follows from this that anything which, like matter, is a being by aggregation, must presuppose true unities from which it is aggregated:

> I believe that where there are only beings by aggregation, there will not in fact be any real beings; for any being by aggregation presupposes beings endowed with a true unity, because it derives its reality only from that of its constituents. It will therefore have no reality at all if each constituent being is still a being by aggregation, for whose reality we have to find some further basis, which in the same way, if we have to go on searching for it, we will never find. (to Arnauld, 30th April 1687; G II 96; WFPT 123)

If a body is the aggregate of the parts into which it is divided, then its reality consists in the parts alone, and not in their being perceived as one. But since each of these parts is further divided, the argument iterates: the body is a perceived unity and a plurality of parts, but each of these parts is also a perceived unity and a plurality of parts, and so on down. If there are no true unities, then, given infinite division, the reality of body will elude analysis: it will reduce to a pure phenomenon. If, on the other hand, there exist true unities in the body, then the body's reality will reduce to the reality of these, while its unity will consist in their being perceived together. It will then be what Leibniz in his correspondence with De Volder calls a "quasi-substance", a plurality of substances with no substantial unity.[21] Leibniz repeats this argument to De Volder:

> Anything that can be divided into many (already actually existing) things is aggregated from many things, and a thing that is aggregated from many things is not one except in the mind, and has no reality except that which is borrowed from what it contains. From this I then inferred that there are therefore indivisible unities in things, because otherwise there will be no true unity in things and no reality that is not borrowed, which is absurd. For where there is no true unity then there is no true multitude. And where there is no reality except that which is borrowed, there will never be reality, since this must in the end be proper to some subject. (30 June 1704; G II 267)

Now it is important to appreciate that Leibniz does not identify the true unities that he claims must be in body with the various actual parts into which it is divided. Body is divided into parts, but resolved into unities. The unities are presupposed by the nature of a body as an aggregate: its reality must reduce to the reality of its constituents.[22] The actual parts, on the other hand, are the result of a motion in common that is actually instituted in matter. But this does not prevent there from being other motions within this part of matter: in fact, of course, Leibniz argues that there are always such differentiated motions in any part of matter, and this is what results in its being infinitely divided.[23] But this also means that no part of matter can be a true unity. Moreover, because all parts of matter are constantly jostling one another, the divisions of matter differ from one instant to another. This means that no part of matter remains the same—

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21 See for instance Leibniz's letter to De Volder of 19 November 1703, in Lodge 2009, 445.

22 I have given a fuller analysis of Leibniz's notion of *presupposition* in Arthur 2011. If A *presupposes* B, then B *is in* A, and A *contains* B. These are equivalent technical notions of wide-ranging application, for which Leibniz gives a formal treatment. I argue that those things are *constituents* of A that are presupposed in every part of A and are not themselves further resolvable, such as points in a line, or simple substances in matter.

23 Cf. Levey 1999, 144-45: "Adjacent parcels of matter form a cohesive whole in virtue of their sharing a motion in common (*motus conspirans*), but this is consistent with each parcel having a motion of its own that divides and distinguishes it from the others. Also there can be further differing motions within each parcel that distinguish *its* parts." Thus having a motion in common is sufficient to individuate a raindrop, for instance, but does not preclude there being a variety of motions within the drop which, according to the Cartesian criterion, divide it within. Just as a line segment can be divided into further line segments, and these again, without limit, it is not necessary that an infinite division should issue in points or infinitesimals, contra Gregory Brown's assertion: "For that the divisions within matter must finally resolve themselves into infinitesimals or minima is something that seems to be guaranteed by Leibniz's assumption that *every* part of matter is divided to infinity" (Brown 2000, 34). Levey identifies this "folds" model as a third model for Leibnizian infinite division, in addition to his "divided block" and "diminishing pennies", although he later describes both it and the divided block models as "incoherent, given Leibniz's metaphysics of matter" (Levey 1999, 148).

i.e. has exactly the same shape and size—through time. The true unities, on the other hand, are precisely things that remain the same through time.

What then is the connection between the true unities presupposed by the reality of body and the actual parts into which matter is divided? The answer, in short, is motion, and the foundation Leibniz provides for it. As we have seen, the parts of matter are actually distinguished from one another by their motions. Motion, on the other hand, is not fully real, according to Leibniz. Because of the relativity of motion, it is impossible to say to which of several bodies in relative motion it belongs. There must nevertheless be some subject of motion, or all motion will be a pure phenomenon. There must also be some foundation for the real distinction of the differing motions (more accurately, tendencies to motion) that exist at each instant, or else there will be no objective basis for distinguishing the actual parts of matter.[24]

This is where Leibniz's revamped notion of substance comes into play. The argument so far has been that there must be real unities in matter, and also that there must be some principles by which the differing motions in matter at any instant might be distinguished. These desiderata are both satisfied by Leibniz's conception of the unities or substances as beings capable of action, for which it is necessary for them to be repositories of *force*. On the one hand, force is "an attribute from which change follows, whose subject is substance itself" (to De Volder, 3 April 1699; GP II 170; Lodge 2009, 313); on the other, it involves an endeavour or striving (*nisus*), and this is what the reality of motion consists in: "there is nothing real in motion but the momentary state which a force endowed with a striving for change must produce" (*Specimen Dynamicum*, 1695; GM VI 236; WFPT 155). This force is thus the foundation for the motion of any actual part of matter at any instant. It is an *entelechy* in the sense that it remains self-identical through the changes of state that it brings into actuality: it is the real foundation at any instant for the motion individuating the actual part of matter that is its body. The differing tendencies to motion to which the entelechies in matter give rise are what make actual the various parts into which matter is divided at different instants.

24 "For at the present moment of its motion, not only is body in a place commensurate with itself, but it also has an endeavour or striving to change place, so that from its subsequent state follows per se from its present one by the force of nature. Otherwise, … there would be absolutely no distinction between bodies, seeing as in a plenum of mass that is uniform in itself the only means for distinguishing them is with respect to motion" (*On Nature Itself*, GP IV 513; WFPT 218-19).

Leibniz does not much stress the role of his entelechies in making the parts of matter actual, but it is there if you look for it. We have already quoted above his riposte to De Volder's claim that the parts of matter could be distinguished only modally: "Nor do the parts of matter differ only modally if they are divided by souls and entelechies, which always persist." (G II 282) There is also this passage:

> Since, therefore, primitive entelechies are dispersed everywhere throughout matter—which can easily be shown from the fact that principles of motion are dispersed throughout matter—the consequence is that souls also are dispersed everywhere throughout matter. (GP VII 329; Russell 1900, 258)

Thus matter is actually divided by its different motions; each of these presupposes a principle, that is, a substance that is the subject of the changes, and a force that results in the changes occurring. Because each part of matter is further divided, there are such substances or true unities (Greek: *monada*) everywhere:

> If there were no divisions of matter in nature, there would not be any diverse things, or rather there would be nothing but the mere possibility of things: but the actual division in masses makes distinct the things that appear, and presupposes (*supponit*) simple substances. (unsent draft to De Volder, 1704-5; GP II 276)

> Since monads or principles of substantial unity are everywhere in matter, it follows from this that there is also an actual infinity, since there is no part, or part of a part, which does not contain monads. (to Des Bosses, 14 Feb 1706; GP II 301; LDB 25; Russell 1900, 129)

This, then, is Leibniz's argument for the actually infinite plurality of monads or simple substances. Because monads are presupposed in every actual part of matter, and matter is infinitely divided, there are actually infinitely many monads. Moreover, this actual infinite is understood syncategorematically, in perfect agreement with his mathematics of the infinite: their multiplicity is greater than any given number:

> In actuals, there is nothing but discrete quantity, namely the multiplicity of monads or simple substances, which is greater than any number whatever in any aggregate whatever that corresponds to the phenomena. (to De Volder, 19th January, 1706; GP II 282)

Thus we have the following contrast between finite aggregates and infinite ones. A finite aggregate is the whole formed by its parts, by analogy with the formation of the natural numbers from unities: $1 + 1 + 1 = 3$. Addition, however, is a mental operation: numbers as such are ideal, as are flocks interpreted as entities distinct from their members. One can imagine an infinite aggregate as similarly corresponding to an infinite sum, as the whole formed by an infinite addition: $1 + 1 + 1 + \ldots = \infty$. But the idea of an infinite whole leads to contradiction, since then a part will be equal to the whole. Thus it is a fiction: while one can work with infinite sums of infinitely small elements under certain well defined conditions without falling into error, there is no such thing in actuality as an infinite sum or an infinite addition: the infinite is not a true whole, and there is no such thing as an infinite number.

In conclusion: according to Leibniz, to say that there are actually infinitely many parts of matter at each instant is to say that there are so many that for any finite number one assigns, there are more. But each of these actual parts presupposes true unities. These constitute what is real about the bodies, since the reality of the aggregate reduces to the reality of its constituents, while the unity of the aggregate is supplied by a perceiving mind. Therefore, "since there is no part, or part of a part, which does not contain monads", bodies are infinite aggregates of monads: in any body there are more monads than can be assigned. For Leibniz, there are actually infinite aggregates, but—in contrast to Cantor—there are no infinite numbers.

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